Gyrokinetic simulations of magnetic fusion plasmas

Tutorial 1

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Simulations of turbulent transport in tokamak plasmas

Context:

- ITER (International Thermonuclear Experimental Reactor)
- Mathematical tool: Vlasov-Maxwell system + gyrokinetic derivation
- Gyrokinetic codes: Ex. GYSELA
Outline

1. Introduction to fusion
2. Why plasma turbulence simulations?
3. How to model plasma?
4. Kinetic theory
Developing a gyrokinetic code ...

▶ Will not be possible with a strong collaboration between physicists, mathematicians and computer scientists

▶ A great acknowledge to all the collaborators

   ▶ Physicists: J. Abiteboul, S. Allfrey, G. Dif-pradalier†, X. Garbet, Ph. Ghendrih, Y. Sarazin, A. Strugarek

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This tutorial will be a mix between physics, mathematics and High Performance Computing
1. A brief introduction to magnetic plasma fusion

2. Plasma turbulence is a subject of utmost importance
Plasma state

- Deuterium-Tritium reaction: the most accessible fusion reaction

\[ D + T \rightarrow He + n + 17.6\text{MeV} \]

- To overcome the electrostatic repulsion, the nuclei must have temperatures > hundred million degrees

- At such temperatures:
  - electrons completely detached from the nucleus
  - the gas is composed of positively (ions) and negatively (electrons) charged particles ⇒ Plasma

- Due to the presence of these charge carriers the plasma is electrically conductive so that it responds strongly to electromagnetic fields
Charged particle motion in a field $\mathbf{B}$

- a strong magnetic field confines the motion of the plasma particles perpendicular to the magnetic field lines to gyro-orbits.

- Parallel to the field lines, the particles move more or less freely (up to magnetic mirror effects).
How to insure confinement along magnetic field line?

- To avoid losses at the ends of the magnetic field, the field lines are usually bent to a torus.

- Plasmas in purely toroidal magnetic fields are subject to drifts that prevent a stable confinement.
Rotational transform

- This problem is solved by a twisting of the magnetic field lines, i.e. the creation of an additional poloidal component of the magnetic field.

- Drift is compensated and vanishes in average.
Stellerator configuration

- twisted magnetic field needed for confinement completely generated by the external field coils

*Wendelstein 7-X in construction at Greifswald in Germany*

- **Difficulties:** Extremely complex geometry, construction is delicate
Tokamak configuration

- Set of external field coils produces a purely toroidal magnetic field
- Additionally, the poloidal magnetic field component is created by a strong toroidal electric current induced in the plasma

The pitch of the field line, i.e. the ratio of toroidal and poloidal revolutions of a field line, is given by the so-called safety factor $q$.

- If $q$ not a rational number, the field line covers a flux surface
- Field lines at $\neq$ radial positions define nested flux surfaces
Most fusion experiments in the world, including ITER now under construction at Cadarache, France, follow Tokamak concept.
How does such a plasma look like in tokamak?

In-vessel visible CCD camera

Tore Supra discharge #42408:

- plasma column $\sim 1\text{m}$
- temperature $\sim 10^6 \, ^\circ\text{C}$

Radiative emission $\equiv$ “cold” edge

\[\text{visualise the magnetic topology}\]
Toroidal geometry and notations

- Notations for the following:
  - \((r, \theta, \phi)\) = (radial, poloidal, toroidal) directions
  - \(a\) minus radius of the torus
  - \(R_0\) major radius of the torus
Lawson criterion

- The condition to obtain a fusion power that is larger than the losses is given by the Lawson criterion

\[ nT \tau_E \geq 3 \times 10^{21} \text{m}^{-3} \text{keV s}^{-1} \]

- To be able to produce energy from fusion reactions, a sufficiently hot \((T)\) and dense \((n)\) plasma must be confined effectively \((\tau_E = \text{confinement time})\)

- Difficulty resides in obtaining the 3 parameters simultaneously
  - Increasing the density by injecting gas into the machine or the temperature by adding additional power to the plasma
    - the confinement tends to deteriorate
  - Particular attention is turn to develop physics scenarios to improve the confinement time \(\tau_E\)
Economic viability of Fusion largely governed by turbulence

- Quality factor $Q$ increases with energy confinement time $\tau_E$

$$Q = \frac{P_{\text{fusion}}}{P_{\text{add}}} \propto \frac{\tau_E}{(\tau_{\text{Lawson}} - \tau_E)}$$

- $\tau_E \sim$ thermal relaxation time, mainly determined by conductive losses \(\Rightarrow\) governed by turbulent transport

- Aim of numerical simulations of plasma turbulence:
  - Predict transport level in present & future devices
  - Open the route towards high confinement regimes
  - Try to understand the physics...
The “engineer” approach: \( \tau_E = \frac{\text{energetic content}}{\text{power losses}} \)

**Energy confinement time \( \tau_E \):**

- A measure of the quality of the confinement
- A basis for extrapolation
  - Semi-empirical scaling law

**Dimensionless parameters:**

- \( \rho^* = \rho_i/a \): required size of the device
- \( \beta = \frac{\text{plasma pressure}}{\text{magnetic pressure}} \)
- \( \nu^* = \text{collisionality of the plasma} \)

- A gap for ITER
  - Uncertainty in prediction
  - Requires understanding physics to validate the extrapolation

\[ \omega_c \tau_E \propto \rho^*^{-3} \beta^{-0.5} \nu^*^{-0.1} \]
Magnetic plasma fusion
How to model plasma?
Plasma kinetic theory
N-body
Kinetic description
Fluid approach

Requires First principle simulations

How to model a thermonuclear plasma?
Phase space in 6D

As shown e.g. by Poincaré, the minimal phase space where all the possible trajectories of a dynamical system are represented is a six-dimensional space:

- 3D in space: \((r, \theta, \varphi)\)
- 3D in velocity: \((v_\parallel, v_\perp, \alpha)\)

**Notation:** \((x, v) \in \mathbb{R}^d \times \mathbb{R}^d\) with \(d = 3\)
**Fields described by Newton-Maxwell’s laws**

- Each charged particles of specie $s$ follows the Newton’s law under the influence of electric and Lorentz forces, i.e:
  \[
  m_s \frac{dv}{dt} = q_s(E + v \times B)
  \]  
  where $B$ is the electric field and $E$ the magnetic field.

- The dynamics of these fields obey Maxwell’s equations:
  \[
  \nabla \cdot E = \rho \quad \text{Gauss} \quad (2)
  \]
  \[
  -\frac{\partial E}{\partial t} + \nabla \times B = j \quad \text{Ampère} \quad (3)
  \]
  \[
  \nabla \cdot B = 0 \quad \text{flux conservation} \quad (4)
  \]
  \[
  \frac{\partial B}{\partial t} + \nabla \times E = 0 \quad \text{Faraday} \quad (5)
  \]
Vector potential and electric potential

According to the flux conservation law \((\nabla \cdot B = 0)\) and \(\nabla \cdot \nabla \times = 0\), there exist a vector potential \(A\) such that:

\[
B = \nabla \times A
\]  (6)

Due to Maxwell-Faraday equation \((\nabla \times E = -\partial_t B)\) and \(\nabla \times \nabla = \vec{0}\), there exist a electric potential \(\phi\) such that:

\[
E = -\nabla \phi - \partial_t A
\]  (7)

where \(A\) and \(\phi\) are defined by taking into account a function \(g\): gauge condition:

\[
A \rightarrow A + \nabla g \quad ; \quad \phi \rightarrow \phi - \partial_t g
\]
Local equation of charge conservation

By taking the divergence of Maxwell-Ampère equation:

\[ \nabla \cdot \nabla \times \mathbf{B} = 0 = \mu_0 \nabla \cdot \mathbf{j} + \epsilon_0 \mu_0 \nabla \cdot \left( \frac{\partial \mathbf{E}}{\partial t} \right) \]

We can write by permuting the spatial and temporal derivatives, then by using the Maxwell-Gauss equation:

\[ \nabla \cdot \left( \frac{\partial \mathbf{E}}{\partial t} \right) = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t} \]

We obtain the local equation of charge conservation:

\[ \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \]
Reduced model ➔ Electrostatic model

- In the electrostatic case, i.e. magnetic field perturbations are not taken into account \( (\partial_t A = 0) \)

- According to equation (7), the electrostatic field is given by

\[
E = -\nabla \phi
\]

- According to Gauss law (2) we obtain the Poisson equation:

\[
-\nabla^2 \phi = \rho
\]
Interactions of the fields with the particles (1/2)

- The interactions of the fields with the particles occurs only through the plasma charge density $\rho(x, t)$ and current $j(x, t)$.

If the plasma consists of particles of species $s$ at positions $(x^i_s, v^i_s)$ for $i = 1, \cdots, N$ then:

- The number density is

$$n_s(x, t) = \sum_{i=1}^{N} \delta(x - x^i_s(t))$$

- The charge density is

$$\rho(x, t) = \sum_s q_s n_s(x, t) \quad (8)$$
Interactions of the fields with the particles (2/2)

- The current density $\mathbf{j}$ is obtained from the mean velocities

$$n_s \mathbf{v}_s(\mathbf{x}, t) = \sum_{i=1}^{N} \mathbf{v}_s^i(t) \, \delta(\mathbf{x} - \mathbf{x}_s^i(t))$$

as

$$\mathbf{j}(\mathbf{x}, t) = \sum_s q_s \mathbf{v}_s(\mathbf{x}, t) \, n_s(\mathbf{x}, t) \quad (9)$$

A complete description of plasma motions is therefore given by the Lorentz force law (Eq. 1) and the Maxwell’s equations (Eqs. 2-5) where the charge density and current density are respectively defined by Eq. (8) and Eq. (9).
Theoretical Hierarchy of plasma physics

- Maxwell equations → well-posed problem
- Model for plasma response? → Several choices
  1. Microscopic description → N-body approach
  2. Kinetic models → Statistical approach
  3. Macroscopic description → Fluid approach
Microscopic description ➞ unrealistic

- A N-body approach would require to solve $N$ coupled equations in 6D phase space for the $N$ particles the system is composed of.

- Knowing that a fusion plasma consists typically of $\sim 10^{20}\, m^{-3}$ ions and electrons

- It is unrealistic to trace all of them even with the most powerful computers in the present day and any foreseeable future.

- Much of the analysis in plasma physics is devoted toward deriving approximate sets of tractable equations
The intermediate models coming in the hierarchy are the kinetic models where instead of solving all the particle motion for each species, a statistical approach is introduced to describe a plasma by a particle distribution function $f_s$.

Although the precise locations of individual particles are lost, detailed knowledge of particle motion is required.

In this sense kinetic theory is still microscopic, even though statistical averages have been employed.
Kinetic description ➔ Vlasov equation (2/2)

- The evolution of $f_s$ is given by Boltzmann equation

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} - \{H_s, f_s\} = C(f_s', f_s) \quad (10)$$

- $\{., .\} =$ Poisson brackets in the canonical coordinates $(q, p)$

- $H_s(q, p)$ Hamiltonian of collisionless single particle motion

$$H_s(q, p) = \frac{1}{2m_s} \left| p - \frac{e_s A}{c} \right|^2 + e_s \phi$$

- For micro-turbulence, a collisionless model is often used ($C(f_s', f_s) = 0$) because:

  collision frequency $\ll$ characteristic frequencies of turbulent fluctuations

- Boltzmann equation $\Rightarrow$ Vlasov equation.
Macroscopic description

- **Fluid approach** consists in reducing even further kinetic theory

- By projecting the kinetic equation (10) on the infinite polynomial velocity basis \( \{1, v, v^2, \ldots, v^k, \ldots\} \) and focuses on the moments of the kinetic equation

  - Other projective basis are possible, especially those based on Hermite or Laguerre polynomials. The convergence is faster in the sense that for a given accuracy as compared to the kinetic model, less moments are needed. However, the physical interpretation of these moments is less clear than with the velocity basis.

- The moment \( \mathcal{M}_k \) of order \( k \) is an integral over velocity space of \( v^{\otimes k} f \), \((\otimes^k \text{ denotes a tensorial product of order } i \leq k) \)
Macroscopic description ⌨ simpler but ... (1/2)

The first moments are the density \((k = 0)\), the flow velocity \((k = 1)\), the pressure \((k = 2)\) and the heat flux \((k = 3)\):

\[
n = \int_{-\infty}^{+\infty} d^3 v \, f \tag{11}
\]
\[
\mathbf{u} = \frac{1}{n} \int_{-\infty}^{+\infty} d^3 v \, \mathbf{v} f \tag{12}
\]
\[
\mathbf{P} \equiv p \mathbf{I} + \tilde{\mathbf{P}} = m \int_{-\infty}^{+\infty} d^3 v \, (\mathbf{v} - \mathbf{u}) \otimes (\mathbf{v} - \mathbf{u}) f \tag{13}
\]
\[
\mathbf{q} = \frac{m}{2} \int_{-\infty}^{+\infty} d^3 v \, |\mathbf{v} - \mathbf{u}|^2 (\mathbf{v} - \mathbf{u}) f \tag{14}
\]
Macroscopic description ➞ simpler but ... (2/2)

- The result of this projection is an infinite set of equations, formally written:

\[
\forall k \in \mathbb{N}, \quad \int d^3v \: v^\otimes k \left[ \text{Eq.((10))} \right] = g(M_{k-1}, M_k, M_{k+1}) \quad (15)
\]

- in which every \(k^{th}\)-order fluid equation couples the order \(k\) and \(k - 1\) moments to the following one \(k + 1\), leading to the infinite fluid hierarchy.

- Should this infinite set of equations be solved, the approach would be equivalent to the kinetic model.
As a greater simplicity is seeked, the fundamental fluid problem lies on how to truncate this infinite hierarchy,

\[ k_c < \infty \], yet retaining the fairest possible amount of physics.

This is referred to as the ‘closure problem’
Fluid approach not sufficient

- Plasmas slightly collisional \(\Rightarrow\) far from the fluid state
- A kinetic approach is necessary
  - Landau resonances, Trapped and fast particles, ...
- Present fluid closures are not sufficient
Magnetic plasma fusion
How to model plasma?
Plasma kinetic theory

N-body
Kinetic description
Fluid approach

Time scale separation ➔ gyrokinetic possible

- Kinetic theory ➔ Solve 6D Vlasov (or Fokker-Planck) equations for each species, coupled to Maxwell equations which involves enormous ranges of spatio-temporal scales

Fast gyromotion ➔ gyrokinetic reduction: $\overline{f}$, $\overline{\phi}$

- Gyroaverage is necessary
- **Gyrokinetic theory** has been developed ➔ Reduce 6D problem to 5D problem

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Plasma kinetic theory

1. Liouville theorem
2. Vlasov-Maxwell system
3. Vlasov equation properties
Liouville theorem

Let us consider a canonical Hamiltonian system:
- \( q = \{q_i\}_{1 \leq i \leq N} \) denote the generalized coordinates,
- \( p = \{p_i\}_{1 \leq i \leq N} \) their conjugate momenta and
- \( H(\{q_i, p_i\}) \) the Hamiltonian.

Let also the phase-space distribution \( D_N(q, p) \) determines the probability \( D_N(q, p) \, d^Nq \, d^Np \) that the system will be found in the infinitesimal volume of phase-space \( d^Nq \, d^Np \).

Then the equilibrium statistical mechanics of such a canonical Hamiltonian system is based on the Liouville theorem,

Liouville theorem: the phase-space distribution function is constant along the trajectories of the system
\[ \Rightarrow D_N \] the density of \( N \) system points in the vicinity of a given system point travelling through phase-space is constant with time.
Liouville theorem proof (1/3)

- Proof comes from the fact that the evolution of $D_N$ is defined by the continuity equation

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \left[ \frac{\partial}{\partial q_i} \left( D_N \frac{dq_i}{dt} \right) + \frac{\partial}{\partial p_i} \left( D_N \frac{dp_i}{dt} \right) \right] = 0 \quad (16)$$

- and that the “velocity field” $(\dot{p}, \dot{q})$ in phase space has zero divergence as a direct consequence of the Hamilton equations of motion

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$
Liouville theorem proof (2/3)

- Indeed, equation (16) can be developed as

\[
\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \left( \frac{\partial D_N}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial D_N}{\partial p_i} \frac{dp_i}{dt} \right) + D_N \sum_{i=1}^{N} \left[ \frac{\partial}{\partial q_i} \left( \frac{dq_i}{dt} \right) + \frac{\partial}{\partial p_i} \left( \frac{dp_i}{dt} \right) \right] = 0
\]

- and according to the previous Hamilton’s relations

\[
\sum_{i=1}^{N} \left[ \frac{\partial}{\partial q_i} \left( \frac{dq_i}{dt} \right) + \frac{\partial}{\partial p_i} \left( \frac{dp_i}{dt} \right) \right] = \sum_{i=1}^{N} \left( \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0
\]

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Then this is equivalent to say that the convective derivative of the density \( \frac{dD_N}{dt} \) is equal to 0, because

\[
\frac{dD_N}{dt} = \frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \left( \frac{\partial D_N}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial D_N}{\partial p_i} \frac{dp_i}{dt} \right) = 0 \quad (17)
\]
BBGKY hierarchy ➤ Boltzmann equation (1/2)

By integrating over part of the variables, the Liouville equation (Eq. 16) can be transform into a chain of equations:

- The first equation connects the evolution of one-particle density probability with the two-particle density probability function and
- Generally the $j$-th equation connects the $j$-th particle density probability function $\mathcal{D}_j = \mathcal{D}_j(q_1 \cdots q_j, p_1 \cdots q_j)$ and the $(j + 1)$-th particle density function.

A truncation of this BBGKY hierarchy of equations is a common starting point for many applications of kinetic theory.
BBGKY hierarchy ➟ Boltzmann equation (2/2)

- In particular, truncation at the first equation or the first two equations can be used to derive classical Boltzmann equations and their first order corrections.

- This derivation is out of the scope of this tutorial, for more details see e.g. [D.Ya Petrina et al., *Mathematical foundations of classical statistical mechanics: continuous system*, 2002]

- In the following, we will focus on the kinetic description of the plasma turbulence and more precisely on the numerical solving of the *Boltzmann equation* and of its collisionless form the *Vlasov equation*. 
Vlasov-Maxwell system for plasma turbulence (1/4)

- In a high temperature fusion plasma with $\sim 10\text{keV}$
  
  collision frequency $\ll$ characteristic frequencies of turbulent fluctuations

- Particles are weakly coupled.
- Multiple particle correlations involving three particles or more are neglected,
- Two particle interaction is reduced to a collision operator for a single particle distribution function
Vlasov-Maxwell system for plasma turbulence (2/4)

Let called $f_s \equiv f_s(x, v, t)$ the 6D function (= the density of particles species $s$ in the phase space $(x, v)$ (3D in space and 3D in velocity) at time $t$), then

- its evolution is governed by the Boltzmann equation

\[
\frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + q_s (E + v \times B) \frac{\partial f_s}{\partial v} = C(f'_s, f_s) \quad (18)
\]

- In the collisionless limit with $C(f'_s, f_s) = 0$, equation (18) yields the Vlasov equation or the collisionless Boltzmann equation

\[
\frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + q_s (E + v \times B) \frac{\partial f_s}{\partial v} = 0 \quad (19)
\]
Vlasov-Maxwell system for plasma turbulence (3/4)

- The electromagnetic fields, \( E \) and \( B \) are determined by the Maxwell’s equations

\[
\nabla \cdot E = \sum_s q_s n_s \tag{20}
\]

\[
- \frac{\partial E}{\partial t} + \nabla \times B = \sum_s j_s \tag{21}
\]

\[
\nabla \cdot B = 0 \tag{22}
\]

\[
\frac{\partial B}{\partial t} + \nabla \times E = 0 \tag{23}
\]
Vlasov-Maxwell system for plasma turbulence (4/4)

Where the source terms, the charge density $n_s(x, t)$ and the current density $j_s(x, t)$ are obtained by taking the velocity moments of $f_s$ as

$$n_s(x, t) = \sum_s q_s \int f_s(x, v, t) \, dv$$

(24)

$$j_s(x, t) = \sum_s q_s \int f_s(x, v, t) \, v \, dv$$

(25)

The Vlasov-Maxwell system, equations (19)-(25), gives a basic description of a high temperature collisionless plasma.

The Vlasov-Poisson model is an approximation of the Vlasov-Maxwell system where the time fluctuations of the magnetic field $\mathbf{B}$ are neglected.
Vlasov equations properties

1. Advective form
2. Conservative forms
Adveective form of Vlasov equation

Let notes $Z = \{x, v\}$ the 6D phase space vector and $\nabla$ the 6D phase-space derivative defined as

$$\nabla_{(x,v)} = \{\nabla_x, \nabla_v\} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial v}\right\} = \frac{\partial}{\partial Z} \tag{26}$$

Then the Vlasov equation (19) can be written as an advection equation in phase-space, with $f : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ (for $d = 6$)

$$\frac{\partial}{\partial t} f(Z, t) + U(Z, t) \cdot \nabla_{(x,v)} f(Z, t) = 0 \tag{27}$$

where $U : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is the 6D phase-space flow.
Definition of characteristics

That means, $U$ is the total time derivative of $Z$,

$$U(Z, t) = \{U_x, U_v\} = \frac{dZ}{dt} = \left\{ \frac{dx}{dt}, \frac{dv}{dt} \right\} = \{v, E + v \times B\} \quad (28)$$

Let now consider the differential system

$$\frac{dZ}{dt} = U(Z(t), t) \quad (29)$$

$$Z(s) = z \quad (30)$$

which is naturally associated to the advection equation (27).

- The solution of the equation (29) are called the characteristics of the advection equation (27).
- Let notes $Z(t; z, s)$ the solution of (29)-(30)
Useful characteristic properties (1/2)

For existence, uniqueness and regularity of the solutions of the previous differential equations (29)-(30), there exists the following classic theorem of theory of differential equations – which proof can be found in e.g. [Amman Book 1990]

**Theorem**

Let assume \( U \in C^{k-1}(\mathbb{R}^d \times [0, T]), \nabla U \in C^{k-1}(\mathbb{R}^d \times [0, T]) \) and for \( \kappa \geq 1 \) that

\[
|U(Z, t)| \leq \kappa(1 + |z|) \quad \forall t \in [0, T] \quad \forall z \in \mathbb{R}^d
\]

then \( \forall s \in [0, T] \) and \( z \in \mathbb{R}^d \), there exists a unique solution \( Z \in C^k([0, T] \times \mathbb{R}^d \times [0, T]) \) of equations (29)-(30).
Useful characteristic properties (2/2)

1. \( \forall t_1, t_2, t_3 \in [0, T] \) and \( z \in \mathbb{R}^d \),

\[
Z(t_3; Z(t_2; z, t_1), t_2) = Z(t_3; z, t_1)
\]

2. \( \forall (t, s) \in [0, T]^2 \), the application \( z \mapsto Z(t; z, s) \) is a \( C^1 \)-diffeomorphism of \( \mathbb{R}^d \) with inverse \( y \mapsto Z(s; y, t) \).

3. The jacobian \( J(t; z, s) = \nabla_z Z(t; z, s) \), i.e.

\[
J(t; 1, s) = \det(\nabla_z Z(t; z, s))
\]

satisfies \( J > 0 \) and

\[
\frac{\partial J}{\partial t} = (\nabla \cdot U)(Z(t; z, s)) J
\]

In particular, if \( \nabla \cdot U = 0 \), \( J(t; 1, s) = J(s; 1, s) = \det \mathbb{I}_d = 1 \) where \( \mathbb{I}_d \) is the identity matrix of order \( d \).