Gyrokinetic simulations of magnetic fusion plasmas

Tutorial 1

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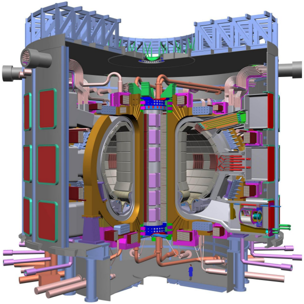
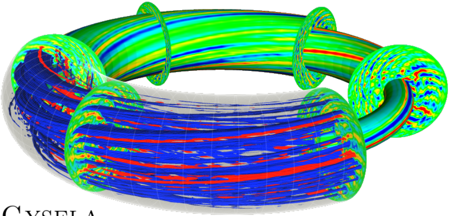
Simulations of turbulent transport in tokamak plasmas

- Context:
  - ITER (International Thermonuclear Experimental Reactor)
  - Mathematical tool : Vlasov-Maxwell system + gyrokinetic derivation
  - Gyrokinetic codes : Ex. GYSELA





GYSELA

Outline

1. Introduction to fusion
2. Why plasma turbulence simulations ?
3. How to model plasma ?
4. Kinetic theory

Developing a gyrokinetic code ...

- Will not be possible with a strong collaboration between physicists, mathematicians and computer scientists
- A great acknowledge to all the collaborators
  - Physicists: J. Abiteboul, S. Allfrey, G. Dif-pradalier†, X. Garbet, Ph. Ghendrih, Y. Sarazin, A. Strugarek
  - Mathematicians:(†† and †††) J.P. Braeunig, N. Crouseilles, M. Mehrenberger, E. Sonnendrücker
  - Computer scientists: Ch. Passeron, G. Latu

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This tutorial will be a mix between physics, mathematics and High Performance Computing

1. A brief introduction to magnetic plasma fusion
2. Plasma turbulence ➠ a subject of utmost importance

Plasma state

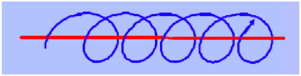
- Deuterium-Tritium reaction: the most accessible fusion reaction

$$D + T \rightarrow He + n + 17.6MeV$$

- To overcome the electrostatic repulsion, the nuclei must have temperatures > hundred million degrees
- At such temperatures:
  - electrons completely detached from the nucleus
  - the gas is composed of positively (ions) and negatively (electrons) charged particles $\Rightarrow$ **Plasma**
- Due to the presence of these charge carriers the plasma is electrically conductive so that it responds strongly to electromagnetic fields

Charged particle motion in a field **B**

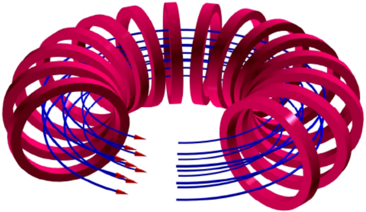
- a strong magnetic field confines the motion of the plasma particles perpendicular to the magnetic field lines to gyro-orbits



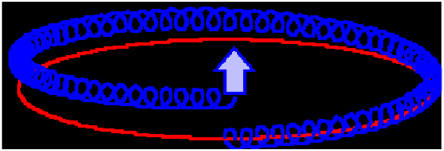
- Parallel to the field lines, the particles move more or less freely (up to magnetic mirror effects)

How to insure confinement along magnetic field line ?

- To avoid losses at the ends of the magnetic field, the field lines are usually bent to a torus

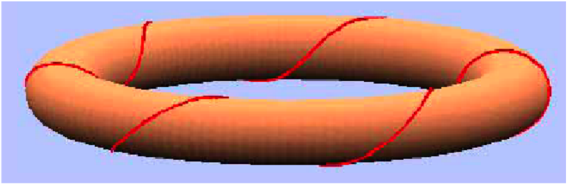


- Plasmas in purely toroidal magnetic fields are subject to drifts that prevent a stable confinement

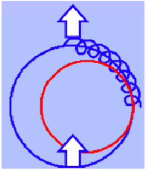


Rotational transform

- This problem is solved by a twisting of the magnetic field lines, i.e. the creation of an additional poloidal component of the magnetic field

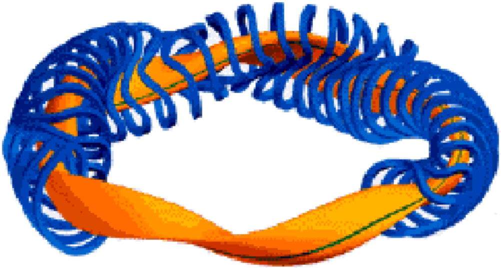


- Drift is compensated and vanishes in average



Stellerator configuration

- twisted magnetic field needed for confinement completely generated by the external field coils

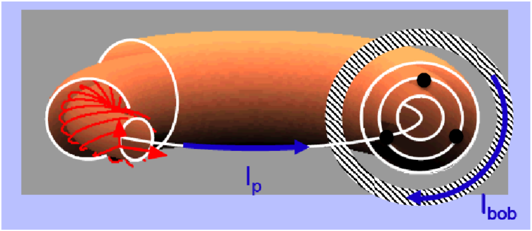


Wendelstein 7-X in construction at Greifswald in Germany

- <u>Difficulties:</u> Extremely complex geometry, construction is delicate

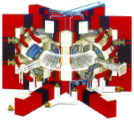
Tokamak configuration

- Set of external field coils produces a purely toroidal magnetic field
- Additionally, the poloidal magnetic field component is created by a strong toroidal electric current induced in the plasma



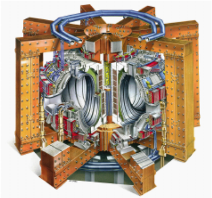
- The pitch of the field line, i.e. the ratio of toroidal and poloidal revolutions of a field line, is given by the so-called safety factor q.
  - If q not a rational number, the field line covers a flux surface
  - Field lines at $\neq$ radial positions define nested flux surfaces

- Most fusion experiments in the world, including ITER now under construction at Cadarache, France, follow Tokamak concept



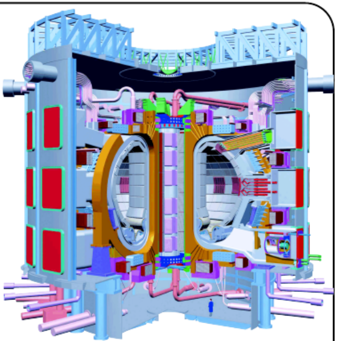
Tore Supra

R=2.4 m
a=0.8 m



JET

R=3.0 m
a=1.2 m



ITER (2016)

R=6.0 m
a=2 m

V=840 m^3

How does such a plasma look like in tokamak ?

In-vessel visible CCD camera

Tore Supra discharge #42408:

- plasma column $\sim$ 1m
- temperature $\sim 10^6$ °C

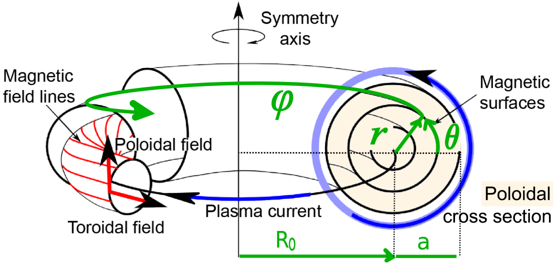
[courtesy J. Gunn]

(Loading film)

Radiative emission $\equiv$ "cold" edge

➠ visualise the magnetic topology

Toroidal geometry and notations



- Notations for the following:
  - (r, θ, ϕ) = (radial,poloidal,toroidal) directions
  - a minus radius of the torus
  - R_0 major radius of the torus

Lawson criterion

- The condition to obtain a fusion power that is larger than the losses is given by the Lawson criterion

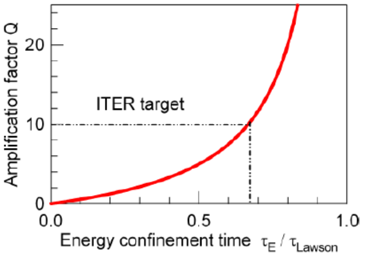
$$nT\tau_E \geq 3 \times 10^{21} m^{-3} keV\ s^{-1}$$

- To be able to produce energy from fusion reactions, a sufficiently hot (T) and dense (n) plasma must be confined effectively (τ_E = confinement time)
- Difficulty resides in obtaining the 3 parameters simultaneously
  - Increasing the density by injecting gas into the machine or the temperature by adding additional power to the plasma
    ➠ the confinement tends to deteriorate
  - Particular attention is turn to develop physics scenarios to improve the confinement time τ_E

Economic viability of Fusion largely governed by turbulence

- Quality factor Q increases with energy confinement time τ_E

$$Q = \frac{P_{\text{fusion}}}{P_{\text{add}}} \propto \frac{\tau_E}{(\tau_{\text{Lawson}} - \tau_E)}$$



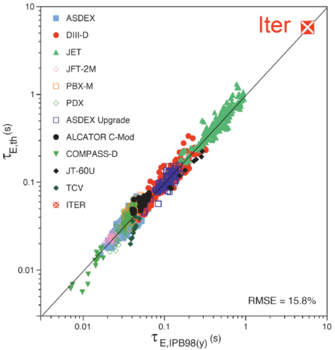
- $\tau_E \sim$ thermal relaxation time, mainly determined by conductive losses ➠ governed by turbulent transport

- Aim of numerical simulations of plasma turbulence:
  - Predict transport level in present & future devices
  - Open the route towards high confinement regimes
  - Try to understand the physics...

The "engineer" approach : $\tau_E = \dfrac{\text{energetic content}}{\text{power losses}}$

Energy confinement time τ_E:

- A measure of the quality of the confinement
- A basis for extrapolation
  ➠ Semi-empirical scaling law



Dimensionless parameters:

- $\rho^* = \rho_i/a$: required size of the device
- β = plasma pressure/magnetic pressure
- ν^* = collisionality of the plasma

$$\omega_c \tau_E \propto \rho_\star^{-3} \beta^{-0.5} \nu_\star^{-0.1}$$

➥ A gap for ITER ➠ Uncertainty in prediction

➥ Requires understanding physics to validate the extrapolation

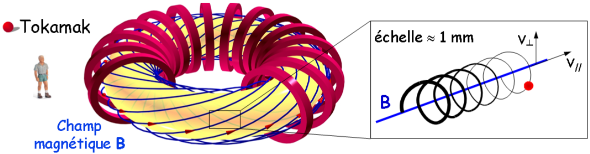
➥ Requires First principle simulations

$$\Downarrow$$

How to model a thermonuclear plasma ?

Phase space in 6D

- As shown *e.g.* by Poincaré, the minimal phase space where all the possible trajectories of a dynamical system are represented is a six-dimensional space
  - 3D in space : (r, θ, φ)
  - 3D in velocity : $(v_{\|}, v_{\perp}, \alpha)$



- Notation: $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^d \times \mathbb{R}^d$ with $d = 3$

Fields described by Newton-Maxwell's laws

- Each charged particles of specie s follows the Newton's law under the influence of electric and Lorentz forces, i.e:

$$m_s \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q_s(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

where $\mathbf{B}$ is the electric field and $\mathbf{E}$ the magnetic field.

- The dynamics of these fields obey Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho \quad \text{Gauss} \tag{2}$$

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mathbf{j} \quad \text{Ampère} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{flux conservation} \tag{4}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad \text{Faraday} \tag{5}$$

Vector potential and electric potential

- According to the flux conservation law ($\nabla \cdot \mathbf{B} = 0$) and $\nabla \cdot \nabla \times = 0$, there exist a vector potential $\mathbf{A}$ such that :

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6}$$

- Due to Maxwell-Faraday equation ($\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$) and $\nabla \times \nabla = \vec{0}$, there exist a electric potential ϕ such that:

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A} \tag{7}$$

- where $\mathbf{A}$ and ϕ are defined by taking into account a function g: gauge condition:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla g \quad ; \quad \phi \rightarrow \phi - \partial_t g$$

Local equation of charge conservation

By taking the divergence of Maxwell-Ampère equation:

$$\nabla \cdot \nabla \times \mathbf{B} = 0 = \mu_0 \nabla \cdot \mathbf{j} + \epsilon_0 \mu_0 \nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right)$$

We can write by permuting the spatial and temporal derivatives, then by using the Maxwell-Gauss equation:

$$\nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right) = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t}$$

We obtain the local equation of charge conservation:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

Reduced model ➠ Electrostatic model

- In the electrostatic case, i.e magnetic field perturbations are not taken into account ($\partial_t \mathbf{A} = 0$)
- According to equation (7), the electrostatic field is given by

$$\mathbf{E} = -\nabla \phi$$

- According to Gauss law (2) we obtain the Poisson equation:

$$-\nabla^2 \phi = \rho$$

Interactions of the fields with the particles (1/2)

- The interactions of the fields with the particles occurs only through the plasma charge density $\rho(\mathbf{x}, t)$ and current $\mathbf{j}(\mathbf{x}, t)$.

If the plasma consists of particles of species s at positions $(\mathbf{x}_s^i, \mathbf{v}_s^i)$ for $i = 1, \cdots, N$ then:

- The number density is

$$n_s(\mathbf{x}, t) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_s^i(t))$$

- The charge density is

$$\rho(\mathbf{x}, t) = \sum_{s} q_s n_s(\mathbf{x}, t) \tag{8}$$

Interactions of the fields with the particles (2/2)

- The current density $\mathbf{j}$ is obtained from the mean velocities

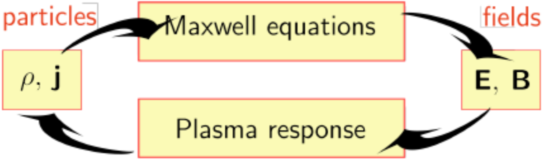
$$n_s \mathbf{v}_s(\mathbf{x}, t) = \sum_{i=1}^{N} \mathbf{v}_s^i(t)\, \delta(\mathbf{x} - \mathbf{x}_s^i(t))$$

as

$$\mathbf{j}(\mathbf{x}, t) = \sum_s q_s \mathbf{v}_s(\mathbf{x}, t)\, n_s(\mathbf{x}, t) \tag{9}$$

A complete description of plasma motions is therefore given by the Lorentz force law (Eq. 1) and the Maxwell's equations (Eqs. 2-5) where the charge density and current density are respectively defined by Eq. (8) and Eq. (9).

Theoretical Hierarchy of plasma physics



- Maxwell equations ➠ well-posed problem
- Model for plasma response ? ➠ Several choices
  1. Microscopic description $\Rightarrow$ N-body approach
  2. Kinetic models $\Rightarrow$ Statistical approach
  3. Macroscopic description $\Rightarrow$ Fluid approach

Microscopic description ➠ unrealistic

- A N-body approach would require to solve N coupled equations in 6D phase space for the N particles the system is composed of.

- Knowing that a fusion plasma consists typically of $\sim 10^{20} m^{-3}$ ions and electrons

- It is unrealistic to trace all of them even with the most powerful computers in the present day and any foreseeable future.

- Much of the analysis in plasma physics is devoted toward deriving approximate sets of tractable equations

Kinetic description ➠ Vlasov equation (1/2)

- The intermediate models coming in the hierarchy are the kinetic models where instead of solving all the particle motion for each species, a statistical approach is introduced to describe a plasma by a particle distribution function f_s.
- Although the precise locations of individual particles are lost, detailed knowledge of particle motion is required.
- In this sense kinetic theory is still microscopic, even though statistical averages have been employed.

Kinetic description ➠ Vlasov equation (2/2)

- The evolution of f_s is given by Boltzmann equation

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} - \{H_s, f_s\} = \mathcal{C}(f_{s'}, f_s) \tag{10}$$

  - $\{.,.\}$ = Poisson brackets in the canonical coordinates $(\mathbf{q}, \mathbf{p})$
  - $H_s(\mathbf{q}, \mathbf{p})$ Hamiltonian of collisionless single particle motion

$$H_s(\mathbf{q}, \mathbf{p}) = \frac{1}{2m_s}\left|\mathbf{p} - \frac{e_s}{c}\mathbf{A}\right|^2 + e_s\phi$$

- For micro-turbulence, a collisionless model is often used ($\mathcal{C}(f_{s'}, f_s) = 0$) because:

$$\textit{collision frequency} \ll \begin{array}{l}\textit{characteristic frequencies}\\ \textit{of turbulent fluctuations}\end{array}$$

- Boltzmann equation $\Rightarrow$ Vlasov equation.

Macroscopic description

- Fluid approach consists in reducing even further kinetic theory
- By projecting the kinetic equation (10) on the infinite polynomial velocity basis $\{1, v, v^2, \ldots, v^k, \ldots\}$ and focuses on the moments of the kinetic equation
  - Other projective basis are possible, especially those based on Hermite or Laguerre polynomials. The convergence is faster in the sense that for a given accuracy as compared to the kinetic model, less moments are needed. However, the physical interpretation of these moments is less clear than with the velocity basis.
- The moment $\mathfrak{M}_k$ of order k is an integral over velocity space of $\mathbf{v}^{\otimes k} f$, ($^{\otimes k}$ denotes a tensorial product of order $i \leq k$)

Macroscopic description ➠ simpler but ... (1/2)

- The first moments are the density ($k=0$), the flow velocity ($k=1$), the pressure ($k=2$) and the heat flux ($k=3$):

$$n = \int_{-\infty}^{+\infty} d^3v\, f \tag{11}$$

$$\mathbf{u} = \frac{1}{n}\int_{-\infty}^{+\infty} d^3v\, \mathbf{v} f \tag{12}$$

$$\bar{\bar{\mathbf{P}}} \equiv p\mathbb{I} + \bar{\bar{\mathbf{\Pi}}} = m\int_{-\infty}^{+\infty} d^3v\, (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u})\, f \tag{13}$$

$$\mathbf{q} = \frac{m}{2}\int_{-\infty}^{+\infty} d^3v\, |\mathbf{v}-\mathbf{u}|^2\, (\mathbf{v}-\mathbf{u})\, f \tag{14}$$

Macroscopic description ➠ simpler but ... (2/2)

- The result of this projection is an infinite set of equations, formally written:

$$\forall k \in \mathbb{N}, \quad \int d^3v\, \mathbf{v}^{\otimes k} \left[\text{Eq.}((10)) \right] = g(\mathfrak{M}_{k-1}, \mathfrak{M}_k, \mathfrak{M}_{k+1}) \quad (15)$$

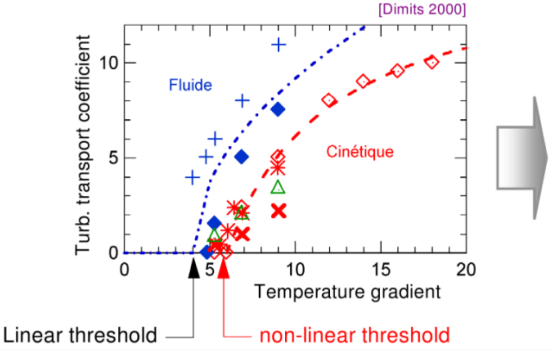
- in which every k^{th}–order fluid equation couples the order k and $k-1$ moments to the following one $k+1$, leading to the infinite fluid hierarchy.
- Should this infinite set of equations be solved, the approach would be equivalent to the kinetic model.

Macroscopic description ➠ Problem of closure

- As a greater simplicity is seeked, the fundamental fluid problem lies on how to truncate this infinite hierarchy,
- *i.e.* close the set of Eqns.(15) at a finite order $k_c < \infty$, yet retaining the fairest possible amount of physics.
- This is referred to as the 'closure problem'

Fluid approach not sufficient

- Plasmas slighly collisional ⇒ far from the fluid state
- A kinetic approach is necessary
  - Landau resonances, Trapped and fast particles, ...
- Present fluid closures are not sufficient

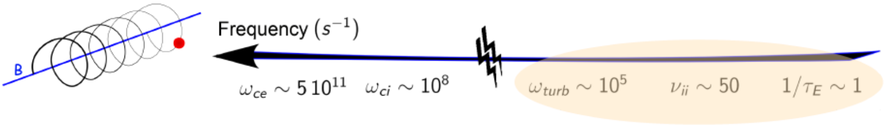


- Large dispersion
- Fluid over estimates transport level
- Non linear threshold in kinetics

Time scale separation ➠ gyrokinetic possible

- Kinetic theory ➠ Solve 6D Vlasov (or Fokker-Planck) equations for each species, coupled to Maxwell equations which involves enormous ranges of spatio-temporal scales

Fast gyromotion ➠ gyrokinetic reduction: $\bar{f}$, $\bar{\phi}$



- gyroaverage is necessary
- Gyrokinetic theory has been developed
  ➠ Reduce 6D problem to 5D problem

Plasma kinetic theory

1. Liouville theorem
2. Vlasov-Maxwell system
3. Vlasov equation properties

Liouville theorem

- Let us consider a canonical Hamiltonian system:
  - $\mathbf{q} = \{\mathbf{q}_i\}_{1\leq i\leq N}$ denote the generalized coordinates,
  - $\mathbf{p} = \{\mathbf{p}_i\}_{1\leq i\leq N}$ their conjugate momenta and
  - $H(\{\mathbf{q}_i, \mathbf{p}_i\})$ the Hamiltonian.
- Let also the phase-space distribution $\mathcal{D}_N(\mathbf{q}, \mathbf{p})$ determines the probability $\mathcal{D}_N(\mathbf{q}, \mathbf{p})\,\mathrm{d}^N q\,\mathrm{d}^N p$ that the system will be found in the infinitesimal volume of phase-space $\mathrm{d}^N q\,\mathrm{d}^N p$.
- Then the equilibrium statistical mechanics of such a canonical Hamiltonian system is based on the Liouville theorem,
- Liouville theorem: the phase-space distribution function is constant along the trajectories of the system

  $\Rightarrow \mathcal{D}_N$ the density of N system points in the vicinity of a given system point travelling through phase-space is constant with time.

Liouville theorem proof (1/3)

- Proof comes from the fact that the evolution of $\mathcal{D}_N$ is defined by the continuity equation

$$\frac{\partial \mathcal{D}_N}{\partial t} + \sum_{i=1}^{N} \left[\frac{\partial}{\partial \mathbf{q}_i} \left(\mathcal{D}_N \frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} \right) + \frac{\partial}{\partial \mathbf{p}_i} \left(\mathcal{D}_N \frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} \right) \right] = 0 \qquad (16)$$

- and that the "velocity field" $(\dot{\mathbf{p}}, \dot{\mathbf{q}})$ in phase space has zero divergence as a direct consequence of the Hamilton equations of motion

$$\frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}_i}$$
$$\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}_i}$$

Liouville theorem proof (2/3)

- Indeed, equation (16) can be developed as

$$\frac{\partial \mathcal{D}_N}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \mathcal{D}_N}{\partial \mathbf{q}_i} \frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} + \frac{\partial \mathcal{D}_N}{\partial \mathbf{p}_i} \frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} \right) + \mathcal{D}_N \sum_{i=1}^{N} \left[\frac{\partial}{\partial \mathbf{q}_i} \left(\frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} \right) + \frac{\partial}{\partial \mathbf{p}_i} \left(\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} \right) \right] = 0$$

- and according to the previous Hamilton's relations

$$\sum_{i=1}^{N} \left[\frac{\partial}{\partial \mathbf{q}_i} \left(\frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} \right) + \frac{\partial}{\partial \mathbf{p}_i} \left(\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} \right) \right] = \sum_{i=1}^{N} \left(\frac{\partial^2 H}{\partial \mathbf{q}_i \partial \mathbf{p}_i} - \frac{\partial^2 H}{\partial \mathbf{p}_i \partial \mathbf{q}_i} \right) = 0$$

Liouville theorem proof (3/3)

- Then this is equivalent to say that the convective derivative of the density $\mathrm{d}\mathcal{D}_N / \mathrm{d}t$ is equal to 0, because

$$\frac{\mathrm{d}\mathcal{D}_N}{\mathrm{d}t} = \frac{\partial \mathcal{D}_N}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \mathcal{D}_N}{\partial \mathbf{q}_i} \frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} + \frac{\partial \mathcal{D}_N}{\partial \mathbf{p}_i} \frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} \right) = 0 \qquad (17)$$

BBGKY hierarchy ➠ Boltzmann equation (1/2)

- By integrating over part of the variables, the Liouville equation (Eq. 16) can be transform into a chain of equations
  - the first equation connects the evolution of one-particle density probability with the two-particle density probability function and
  - generally the j-th equation connects the j-th particle density probability function $\mathcal{D}_j = \mathcal{D}_j(\mathbf{q}_1 \cdots \mathbf{q}_j, \mathbf{p}_1 \cdots \mathbf{q}_j)$ and the $(j+1)$-th particle density function.
- A truncation of this BBGKY hierarchy of equations is a common starting point for many applications of kinetic theory.

BBGKY hierarchy ➠ Boltzmann equation (2/2)

- In particular, truncation at the first equation or the first two equations can be used to derive classical Boltzmann equations and their first order corrections.
- This derivation is out of the scope of this tutorial, for more details see *e.g.* [D.Ya Petrina et al., *Mathematical foundations of classical statistical mechanics: continuous system*, 2002]
- In the following, we will focus on the kinetic description of the plasma turbulence and more precisely on the numerical solving of the **Boltzmann equation** and of its collisionless form the **Vlasov equation**.

Vlasov-Maxwell system for plasma turbulence (1/4)

- In a high temperature fusion plasma with $\sim 10keV$

$$\text{collision frequency} \ll \begin{array}{l}\text{characteristic frequencies}\\ \text{of turbulent fluctuations}\end{array}$$

- Particles are weakly coupled.
- Multiple particle correlations involving three particles or more are neglected,
- Two particle interaction is reduced to a collision operator for a single particle distribution function

Vlasov-Maxwell system for plasma turbulence (2/4)

Let called $f_s \equiv f_s(\mathbf{x}, \mathbf{v}, t)$ the 6D function (= the density of particles species s in the phase space $(\mathbf{x}, \mathbf{v})$ (3D in space and 3D in velocity) at time t), then

- its evolution is governed by the Boltzmann equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + q_s \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \frac{\partial f_s}{\partial \mathbf{v}} = C(f_{s'}, f_s) \qquad (18)$$

- In the collisionless limit with $C(f_{s'}, f_s) = 0$, equation (18) yields the Vlasov equation or the collisionless Boltzmann equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + q_s \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \frac{\partial f_s}{\partial \mathbf{v}} = 0 \qquad (19)$$

Vlasov-Maxwell system for plasma turbulence (3/4)

- The electromagnetic fields, **E** and **B** are determined by the Maxwell's equations

$$\nabla \cdot \mathbf{E} = \sum_s q_s n_s \tag{20}$$

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \sum_s \mathbf{j}_s \tag{21}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{22}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{23}$$

Vlasov-Maxwell system for plasma turbulence (4/4)

- Where the source terms, the charge density $n_s(\mathbf{x}, t)$ and the current density $\mathbf{j}_s(\mathbf{x}, t)$ are obtained by taking the velocity moments of f_s as

$$n_s(\mathbf{x}, t) = \sum_s q_s \int f_s(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \tag{24}$$

$$\mathbf{j}_s(\mathbf{x}, t) = \sum_s q_s \int f_s(\mathbf{x}, \mathbf{v}, t)\, \mathbf{v}\, d\mathbf{v} \tag{25}$$

- The Vlasov-Maxwell system, equations (19)-(25), gives a basic description of a high temperature collisionless plasma.
- The Vlasov-Poisson model is an approximation of the Vlasov-Maxwell system where the time fluctuations of the magnetic field $\mathbf{B}$ are neglected.

Vlasov equations properties

1. Advective form
2. Conservative forms

Advective form of Vlasov equation

Let notes $\mathbf{Z} = \{\mathbf{x}, \mathbf{v}\}$ the 6D phase space vector and ∇ the 6D phase-space derivative defined as

$$\nabla_{(\mathbf{x},\mathbf{v})} = \{\nabla_{\mathbf{x}}, \nabla_{\mathbf{v}}\} = \left\{ \frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{v}} \right\} = \frac{\partial}{\partial \mathbf{Z}} \qquad (26)$$

Then the Vlasov equation (19) can be written as an advection equation in phase-space, with $f : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ (for $d = 6$)

$$\frac{\partial}{\partial t} f(\mathbf{Z}, t) + \mathbf{U}(\mathbf{Z}, t) \cdot \nabla_{(\mathbf{x},\mathbf{v})} f(\mathbf{Z}, t) = 0 \qquad (27)$$

where $\mathbf{U} : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is the 6D phase-space flow.

Definition of characteristics

That means, $\mathbf{U}$ is the total time derivative of $\mathbf{Z}$,

$$\mathbf{U}(\mathbf{Z},t) = \{\mathbf{U_x}, \mathbf{U_v}\} = \frac{d\mathbf{Z}}{dt} = \left\{\frac{d\mathbf{x}}{dt}, \frac{d\mathbf{v}}{dt}\right\} = \{\mathbf{v}, \mathbf{E} + \mathbf{v} \times \mathbf{B}\} \qquad (28)$$

Let now consider the differential system

$$\frac{d\mathbf{Z}}{dt} = \mathbf{U}(\mathbf{Z}(t),t) \qquad (29)$$

$$\mathbf{Z}(s) = \mathbf{z} \qquad (30)$$

which is naturally associated to the advection equation (27).

- The solution of the equation (29) are called the characteristics of the advection equation (27).
- Let notes $\mathbf{Z}(t; \mathbf{z}, s)$ the solution of (29)-(30)

Useful characteristic properties (1/2)

For existence, uniqueness and regularity of the solutions of the previous differential equations (29)-(30), there exists the following classic theorem of theory of differential equations – which proof can be found in *e.g.* *[Amman Book1990]*

Theorem

Let assume $\mathbf{U} \in C^{k-1}(\mathbb{R}^d \times [0,T])$, $\nabla \mathbf{U} \in C^{k-1}(\mathbb{R}^d \times [0,T])$ *and for* $\kappa \geq 1$ *that*

$$|\mathbf{U}(\mathbf{Z},t)| \leq \kappa(1+|\mathbf{z}|) \qquad \forall t \in [0,T] \quad \forall \mathbf{z} \in \mathbb{R}^d$$

then $\forall s \in [0,T]$ *and* $\mathbf{z} \in \mathbb{R}^d$, *there exists a unique solution* $\mathbf{Z} \in C^k([0,T] \times \mathbb{R}^d \times [0,T])$ *of equations (29)-(30).*

Useful characteristic properties (2/2)

1. $\forall t_1, t_2, t_3 \in [0, T]$ and $\mathbf{z} \in \mathbb{R}^d$

$$\mathbf{Z}(t_3; \mathbf{Z}(t_2; \mathbf{z}, t_1), t_2) = \mathbf{Z}(t_3; \mathbf{z}, t_1)$$

2. $\forall (t, s) \in [0, T]^2$, the application $\mathbf{z} \mapsto \mathbf{Z}(t; \mathbf{z}, s)$ is a C^1-diffeomorphism of $\mathbb{R}^d$ with inverse $\mathbf{y} \mapsto \mathbf{Z}(s; \mathbf{y}, t)$.
3. The jacobian $J(t; \mathbf{z}, s) = \nabla_{\mathbf{z}} \mathbf{Z}(t; \mathbf{z}, s)$, i.e. $J(t; 1, s) = \det(\nabla_{\mathbf{z}} \mathbf{Z}(t; \mathbf{z}, s))$ satisfies $J > 0$ and

$$\frac{\partial J}{\partial t} = (\nabla \cdot \mathbf{U})(\mathbf{Z}(t; \mathbf{z}, s))\, J$$

In particular, if $\nabla \cdot \mathbf{U} = 0$, $J(t; 1, s) = J(s; 1, s) = \det \mathbb{I}_d = 1$ where $\mathbb{I}_d$ is the identity matrix of order d.