



B. Després+
X. Blanc
LJLL-Paris
VI+CEA

Thanks to
same
colleagues as
before
plus C. Buet,
H. Egly and
R. Sentis

Numerical methods for FCI

Part IV

Multi-temperature fluid models

Hele-Shaw models

B. Després+ X. Blanc
LJLL-Paris VI+CEA

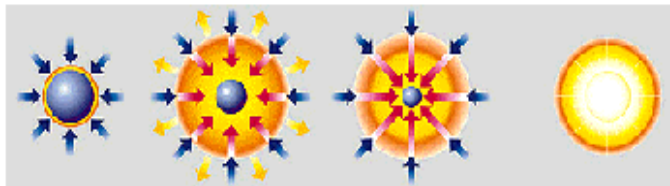
Thanks to same colleagues as before
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Introduction

Numerical discretization of $T_i - T_e$ model

Coupling with radiation

Hele-Shaw models



Irradiation Implosion Ignition Combustion

During the implosion, pure hydrodynamics is a very strong hypothesis.

It is much more relevant to consider multi-temperature models.



Introduction

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discretization
of $T_i - T_e$
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models

- One temperature for ions T_i and one temperature for electrons T_e : $T_i - T_e$ model
Basic considerations
- One temperature for the matter, and one temperature for radiation T_r
- Hele-Shaw model for the stability of the ablation front



The simplified $T_i - T_e$ model

Starting point is (page 9 and page 23 of the notes)

$$\begin{cases} D_t \rho + \rho \nabla \cdot \mathbf{u} & = 0, \\ \rho D_t \mathbf{u} + \nabla p & = \mathbf{F}_r, \\ \rho D_t \varepsilon_e + p_e \nabla \cdot \mathbf{u} - \nabla \cdot (\chi_e \nabla T_e) + W_{ei} & = Q_r + S, \\ \rho D_t \varepsilon_i + p_i \nabla \cdot \mathbf{u} + \nabla \cdot (\chi_i \nabla T_i) - W_{ei} & = 0, \end{cases}$$

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The unknowns of this system are the density $\rho(\mathbf{x}, t) \in \mathbb{R}^+$ of the plasma, its velocity $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$ and its pressure $p(\mathbf{x}, t) \in \mathbb{R}$. We also have electronic and ionic values : pressures $p_e(\mathbf{x}, t), p_i(\mathbf{x}, t) \in \mathbb{R}$ (with $p = p_e + p_i$), energies $\varepsilon_e(\mathbf{x}, t), \varepsilon_i(\mathbf{x}, t) \in \mathbb{R}$ (with $E = E_e + E_i$), and temperatures $T_e(\mathbf{x}, t), T_i(\mathbf{x}, t)$. Here, The terms \mathbf{F}_r and Q_r are the radiative sources, and S is an additional source term modelling the laser energy drop.

This set of equations is closed by an adapted equation of state

$$(\varepsilon_e, p_e, \varepsilon_i, p_i) = \mathcal{F}(\rho, T_e, T_i).$$

We will assume that the fluid is described by a perfect gas EOS

$$p_i = (\gamma_i) \rho C_{vi} T_i = (\gamma_i) \rho \varepsilon_i, \quad \varepsilon_i = C_{vi} T_i$$

and the electronic part is described by a perfect gas EOS

$$p_e = (\gamma_e) \rho C_{ve} T_e = (\gamma_e) \rho \varepsilon_e, \quad \varepsilon_e = C_{ve} T_e.$$

Since electrons are monoatomic

$$\gamma_e = \frac{5}{3}.$$



Hydrodynamics of the $T_i - T_e$ model

The hydrodynamic part is

$$\begin{cases} D_t \rho + \rho \nabla \cdot \mathbf{u} & = 0, \\ \rho D_t \mathbf{u} + \nabla p & = 0, \\ \rho D_t \varepsilon_e + p_e \nabla \cdot \mathbf{u} & = 0, \\ \rho D_t \varepsilon_i + p_i \nabla \cdot \mathbf{u} & = 0, \end{cases}$$

This system is non conservative. For discontinuous functions a and b

the product $a \partial_x b$ is not defined.

What is a shock in such a system ?

We need to transform it into a conservative system of conservation laws.

Universal principles : mass is preserved, momentum is preserved ad total energy is preserved.

We get after convenient manipulations the correct conservative formulation (in 1D)

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p) = 0, \\ \partial_t (\rho e) + \partial_x (\rho u e + p u) = 0, \end{cases}$$

with

$$p = p_i + p_e \text{ and } e = \varepsilon_i + \varepsilon_e + \frac{1}{2} u^2.$$

Question : Is there a fourth conservation laws ?



Conservation of the electronic entropy

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Convenient manipulations show that smooth solutions satisfy

$$\partial_t (\rho(\alpha S_i + \beta S_e)) + \partial_x (\rho u(\alpha S_i + \beta S_e)) = 0, \quad \forall \alpha, \beta.$$

For discontinuous solutions, these relations are not equivalent.

The correct choice is $\alpha = 0$ and $\beta = 1$. This is called the Born-Oppenheimer hypothesis. It is related to the fact that $\frac{m_e}{m_i}$ is small.

Zeldovich-Raizer, Cordier (PhD thesis 96), Degond-Luquin, Massot, ...

Finally

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho S_e) + \partial_x(\rho u S_e) = 0, \\ \partial_t(\rho e) + \partial_x(\rho u e + pu) = 0. \end{cases}$$

The mathematical entropy law writes

$$\partial_t(\rho S_i) + \partial_x(\rho u S_i) \geq 0.$$

As a consequence the ionic entropy increases at shocks while the electronic entropy is constant as shocks

$$S_i^+ > S_i^-, \quad S_e^+ = S_e^-.$$

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Exercise : prove it.

Solution

$$\begin{cases} -\sigma(\rho_R - \rho_L) + (\rho_R u_R - \rho_L u_L) = 0, \\ -\sigma(\rho_R S_{e,R} - \rho_L S_{e,L}) + (\rho_R u_R S_{e,R} - \rho_L u_L S_{e,L}) = 0, \\ -\sigma(\rho_R S_{i,R} - \rho_L S_{i,L}) + (\rho_R u_R S_{i,R} - \rho_L u_L S_{i,L}) > 0. \end{cases}$$

So $\rho_R(u_R - \sigma) = \rho_L(u_L - \sigma)$. This is the constant mass flux $D = \rho_R(u_R - \sigma) = \rho_L(u_L - \sigma)$. Therefore

$$DS_{e,R} = DS_{e,L} \text{ and } DS_{i,R} > DS_{i,L}.$$

Assume a shock and the mass flux is positive $D > 0$. Then

$$S_{e,R} = S_{e,L} \text{ and } S_{i,R} > S_{i,L}.$$

CQFD

This behavior is absolutely fundamental : it explains that ions and electrons behave differently at shocks. In summary **physical considerations** show that is the correct eulerian system of conservation laws to analyze for the two temperature $T_i - T_e$ model.

In Lagrange variable in dimension one, one gets

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m p = 0, \\ \partial_t S_e = 0, \\ \partial_t e + \partial_m(pu) = 0. \end{cases} \quad p = p_i + p_e,$$

Introduction

Numerical discretization of $T_i - T_e$ model

Set

$$\rho^2 c_i^2 = -\frac{\partial p_i}{\partial \tau | S_i}, \quad \rho^2 c_e^2 = -\frac{\partial p_e}{\partial \tau | S_e} \text{ and } \rho^2 c^2 = -\frac{\partial p_i}{\partial \tau | S_i} - \frac{\partial p_e}{\partial \tau | S_e}$$

Coupling with radiation

The sound speed of the lagrangian system is ρc where $c^2 = c_i^2 + c_e^2$.

Hele-Shaw models

The natural Lagrangian scheme is now

$$\begin{cases} \frac{M_j}{\Delta t} (\tau_j^L - \tau_j^n) - u_{j+\frac{1}{2}}^* + u_{j-\frac{1}{2}}^* = 0, \\ \frac{M_j}{\Delta t} (u_j^L - u_j^n) + p_{j+\frac{1}{2}}^* - p_{j-\frac{1}{2}}^* = 0, \\ (S_e)_j^L - (S_e)_j^n = 0, \\ \frac{M_j}{\Delta t} (e_j^L - e_j^n) + p_{j+\frac{1}{2}}^* u_{j+\frac{1}{2}}^* - p_{j-\frac{1}{2}}^* u_{j-\frac{1}{2}}^* = 0, \end{cases}$$

with the solver

$$\begin{cases} u_{j+\frac{1}{2}}^* = \frac{1}{2} (u_j^n + u_{j+1}^n) + \frac{1}{2\rho c} (p_j^n - p_{j+1}^n) \\ p_{j+\frac{1}{2}}^* = \frac{1}{2} (p_j^n + p_{j+1}^n) + \frac{\rho c}{2} (u_j^n - u_{j+1}^n), \\ (\rho c)_{j+\frac{1}{2}} = \frac{1}{2} [(\rho c)_j^n + (\rho c)_{j+1}^n]. \end{cases}$$

A pure Lagrangian scheme is such that $f^{n+1} = f^L$ for all f .



A simplified eulerian $T_i - T_e$ system with source terms writes

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p_i + p_e) = 0 \\ \partial_t \rho \varepsilon_i + \partial_x \rho u \varepsilon_i + p_i \partial_x u = \frac{1}{\tau_{ei}} (T_e - T_i) \\ \partial_t \rho \varepsilon_e + \partial_x \rho u \varepsilon_e + p_e \partial_x u = \frac{1}{\tau_{ei}} (T_i - T_e) + \partial_x (K_e \partial_x T_e). \end{cases}$$

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The relaxation time is τ_{ei} . The electronic diffusion coefficient is K_e . We assume that

$$\varepsilon_i = C_{vi} T_i \text{ et } \varepsilon_e = C_{ve} T_e.$$

The rigorous way to write this is

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p_i + p_e) = 0 \\ \partial_t \rho S_e + \partial_x \rho u S_e = \frac{1}{\tau_{ei} T_e} (T_i - T_e) + \frac{1}{T_e} \partial_x (K_e \partial_x T_e) \\ \partial_t \rho e + \partial_x (\rho u e + p_i u + p_e u) = \partial_x (K_e \partial_x T_e), \end{cases}$$

where the unknowns are the density ρ , the momentum ρu , the electronic entropy ρS_e and the total energy ρe .



Numerical solution base on a splitting strategy

First stage : solve the hydro. Second stage solve the remaining part

Introduction

Numerical discretization of $T_i - T_e$ model

$$\begin{cases} \partial_t \rho = 0 \\ \partial_t \rho u = 0 \\ \partial_t \rho \varepsilon_i = \frac{1}{\tau_{ei}} (T_e - T_i) \\ \partial_t \rho \varepsilon_e = \frac{1}{\tau_{ei}} (T_i - T_e) + \partial_x (K_e \partial_x T_e). \end{cases}$$

Coupling with radiation

We use the linear law

$$\varepsilon_i = C_{vi} T_i \text{ and } \varepsilon_e = C_{ve} T_e.$$

Hele-Shaw models

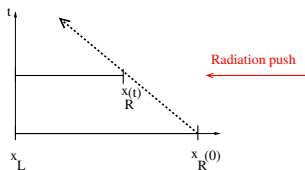
The numerical solution of the system can be computed with an implicit linear solver in case the gas is described by perfect gas equations of state.

$$\begin{cases} \rho^{n+1} & = \rho^L, \\ u^{n+1} & = u^L, \\ \rho^L C_{vi} \frac{(T_i)_j^{n+1} - (T_i)_j^L}{\Delta t} & = \frac{1}{\tau_{ei}} ((T_e)_j^{n+1} - (T_i)_j^{n+1}), \\ \rho^L C_{ve} \frac{(T_e)_j^{n+1} - (T_e)_j^L}{\Delta t} & = \frac{1}{\tau_{ei}} ((T_i)_j^{n+1} - (T_e)_j^{n+1}) \\ & + \frac{K_{e,i+\frac{1}{2}} ((T_e)_{j+1}^{n+1} - (T_e)_j^{n+1}) - K_{e,i-\frac{1}{2}} ((T_e)_j^{n+1} - (T_e)_{j-1}^{n+1})}{\Delta x^2}. \end{cases}$$



An example relevant for ICF in direct drive

The Piston velocity is $W_p > 0$. The domain is $\Omega(t) = \{x_L = 0 < x < x_R(t) = x_R^0 - tW_p\}$.



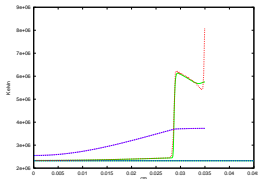
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The ionic temperature is the discontinuous curve. The electronic temperature is the continuous curve. The initial temperature is the blue curve (225 000 K).



The ionic part of the gas is violently heated by the shock. The electronic temperature is continuous everywhere. The temperature relaxation is visible behind the shock. In front of the shock a preheating phenomenon is visible. This calculation shows the great importance of shocks for ICF flows in the context of direct drive.



The Rankine-Hugoniot relation for the $T_i - T_e$ model

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Start from

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p_i + p_e) = 0 \\ \partial_t \rho S_e + \partial_x \rho u S_e = \frac{1}{\tau_{ei} T_e} (T_i - T_e) + \frac{1}{T_e} \partial_x (K_e \partial_x T_e) \\ \partial_t \rho e + \partial_x (\rho u e + p_i u + p_e u) = \partial_x (K_e \partial_x T_e), \end{cases}$$

The Rankine-Hugoniot relations are

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + [\rho u^2 + p_i + p_e] = 0, \\ [T_e] = 0, \\ -\sigma[\rho S_e] + [\rho u S_e] = \frac{1}{T_e} [K_e \partial_x T_e], \\ -\sigma[\rho e] + [\rho u e + p_i u + p_e u] = [K_e \partial_x T_e]. \end{cases}$$

Notice that the continuity of T_e is provided by diffusion operator.

Problem Prove that the continuity of S_e is recovered in the limit

$$K_e \rightarrow 0^+.$$

All numerical results support the conjecture.

Works by Lefloch, Coquel and coworkers (Chalon, Berthon, ...) on a similar problem.



Discretization of boundary conditions

Assume an additional “no-heat flux” physical conditions.

The boundary conditions are

$$x = 0 = x_L, \quad u_{0,t} = 0, \quad \partial_x T_e = 0$$

and

$$x = x_R(t) = x_R(0) - tW_p, \quad u_{x_R(t),t} = -W_p, \quad \partial_x T_e = 0$$

One has 2 boundary conditions and 4 equations

$$\#(\text{bc}) = 2, \quad \#(\text{eq}) = 4.$$

Question : how can we do ?

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There is no problem for the second stage of the algorithm

$$\begin{cases} \partial_t \rho = 0 \\ \partial_t \rho u = 0 \\ \partial_t \rho \varepsilon_i = \frac{1}{\tau_{ei}} (T_e - T_i) \\ \partial_t \rho \varepsilon_e = \frac{1}{\tau_{ei}} (T_i - T_e) + \partial_x (K_e \partial_x T_e). \end{cases}$$

plus homogeneous Neumann conditions $\partial_x T_e = 0$ at boundaries.

So the real problem is for the Lagrangian stage of the algorithm.

We are left with

$$\#(\text{bc}) = 1, \quad \#(\text{eq}) = 4.$$

But it works : this is the miracle of Lagrangian scheme+splitting for this problem !!

Structure of the discrete problem

$$\left\{ \begin{array}{l} \frac{M_j}{\Delta t} (\tau_j^L - \tau_j^n) - u_{j+\frac{1}{2}}^* + u_{j-\frac{1}{2}}^* = 0, \\ \frac{M_j}{\Delta t} (u_j^L - u_j^n) + p_{j+\frac{1}{2}}^* - p_{j-\frac{1}{2}}^* = 0, \\ (S_e)_j^L - (S_e)_j^n = 0, \\ \frac{M_j}{\Delta t} (e_j^L - e_j^n) + p_{j+\frac{1}{2}}^* u_{j+\frac{1}{2}}^* - p_{j-\frac{1}{2}}^* u_{j-\frac{1}{2}}^* = 0, \end{array} \right.$$

$$\#(\text{bc}) = 1, \quad \#(\text{what is needed}) = 2.$$

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Recall the rule : the equations of the discrete Riemann solver are more important than its solution

$$\left\{ \begin{array}{l} p_B - p_L + (\rho c)_L (u_B - u_L) = 0, \\ u_P = -W_P. \end{array} \right. \iff \left\{ \begin{array}{l} p_B = p_L + (\rho c)_L (u_B - u_L), \\ u_P = -W_P. \end{array} \right.$$

So we use in the last $j = J_{\max}$

$$\left\{ \begin{array}{l} p_{j+\frac{1}{2}}^* = p_j + (\rho c)_j (-W_P - u_j), \\ u_{j+\frac{1}{2}}^* = -W_P. \end{array} \right.$$

that we plug into the scheme.

Finally : 1 boundary condition is enough for the hydro !!



A simple grey non equilibrium model

page 33 of the notes, one group

$$E_r = \int_0^\infty E_\nu d\nu$$

plus hydrodynamics. A rigorous justification is possible, but with an "almost" physical scaling.

Consider the simplified model for radiation+matter

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}(\rho) + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(p + p_r) = 0, \\ \frac{\partial}{\partial t}(\rho E + E_r) + \nabla \cdot ((\rho E + E_r) \mathbf{u}) + (p + p_r) \nabla \cdot \mathbf{u} = \nabla \cdot \left(\frac{1}{3\sigma_t} \nabla T_r^4 \right), \\ \frac{\partial}{\partial t} E_r + \nabla \cdot (\mathbf{u} E_r) + p_r \nabla \cdot \mathbf{u} = \nabla \cdot \left(\frac{1}{3\sigma_t} \nabla T_r^4 \right) + \sigma_a (T^4 - T_r^4), \end{array} \right.$$

with the grey hypothesis $\sigma_t = \sigma_a + \sigma_s$ and $p_r = \frac{E_r}{3}$. Fundamental is

$$E_r = a T_r^4, \quad a = \frac{8\pi^5 k^4}{15c^3 h^3} = \text{Stefan-Boltzmann constant.}$$

See Buet+D. for a rigorous justification.

Trick : Define ε_r

$$E_r = \rho \varepsilon_r.$$

The radiation equation rewrites

$$\frac{\partial}{\partial t} \rho \varepsilon_r + \nabla \cdot (\mathbf{u} \rho \varepsilon_r) + p_r \nabla \cdot \mathbf{u} = \nabla \cdot \left(\frac{1}{3\sigma_t} \nabla T_r^4 \right) + \sigma_a (T^4 - T_r^4)$$

and the radiative pressure rewrites

$$p_r = (\gamma_r - 1) \rho \varepsilon_r \quad \text{with } \gamma_r = \frac{4}{3}.$$



Introduction

The first task is to discretize the non equilibrium asymptotic set of equations. Setting $\sigma_a = 0$ and $\sigma_s = +\infty$ then we obtain the simplified hyperbolic set of equations in 1D

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$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho v) = 0, \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p + p_r) = 0, \\ \frac{\partial}{\partial t}(\rho E + E_r) + \frac{\partial}{\partial x}(\rho E v + p v + p_r v) = 0, \\ \frac{\partial}{\partial t} S_r + \frac{\partial}{\partial x}(S_r v) = 0, \end{array} \right. \quad \begin{array}{l} p_r = \frac{E_r}{3}, \\ E_r = T_r^4, \\ S_r = T_r^3. \end{array}$$

The solver is compatible with the 1D Rankine-Hugoniot relations

$$\left\{ \begin{array}{l} -\sigma[\rho] + [\rho v] = 0, \\ -\sigma[\rho v] + [\rho v^2 + p + p_r] = 0, \\ -\sigma[\rho E + E_r] + [\rho v E + p v + p_r v] = 0, \\ -\sigma[S_r] + [v S_r] = 0. \end{array} \right.$$

All this is compatible with the fact that

$$S_r = T_r^3$$

is the number of photons if the radiation is Planckian.

Introduction

The group of photons in direction $\Omega \in S^2$ and with frequency $\nu > 0$ has intensity $I(\nu, \Omega)$

Numerical discretization of $T_i - T_e$ model

$$\vec{v}_{\text{photon}} = c\Omega, \quad e_\nu = h\nu, \quad I(\nu, \Omega) = n_{\text{photons}} e_\nu.$$

Coupling with radiation

The energy of radiation is $E_r = \int_{\nu, \Omega} I d\nu d\Omega$.

The number of photons is $N_r = \int_{\nu, \Omega} \frac{I}{h\nu} d\nu d\Omega$.

The entropy of radiation is $S_r = -\frac{2k}{c^3} \int_{\nu, \Omega} \nu^2 (n \log n - (n+1) \log(n+1)) d\nu d\Omega, \quad n = \frac{I}{\nu^3}$.

Hele-Shaw models

Assume a Planckian distribution with radiative temperature T_r

$$I = \frac{2h}{c^2} \times \frac{\nu^3}{e^{\frac{h\nu}{kT_r}} - 1}.$$

Then dimension analysis shows that : $E_r = \alpha T_r^4$, $N_r = \beta T_r^3$ and $S_r = \gamma T_r^3$.
Therefore S_r is the number of photons.

Read the facinating book by Steven Weinberg : The first three minutes (of universe).

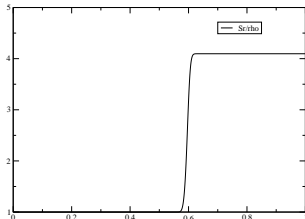
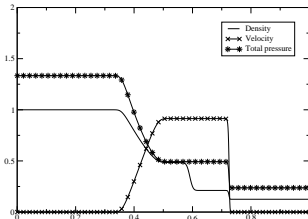
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The first test problem is a radiative Riemann problem. On the left we plot density, velocity and total pressure versus the position x at $t = 0.1$ with 1000 cells : CFL=0.5



On the right. Radiative entropy $s_r = \frac{T_r^3}{\rho} = \frac{S_r}{\rho}$ versus x at $t = 0.1$ with 1000 cells : CFL=0.5. One notices the exact preservation of s_r across the shock and in the rarefaction fan



Numerical comparison with a moment model

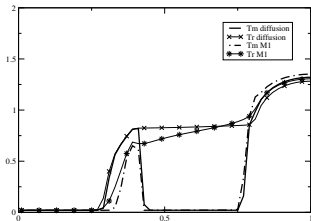
Full system. The 1D solver is implicit.

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100 cells, $T = 0.005$.

Right : T and T_r for diffusion and the variable Eddington factor.

Left : E_r , F_r and $f = \frac{F_r}{E_r}$.

One notices that the non-equilibrium diffusion model overpredicts the propagation of radiation.
It justifies (numerically) higher order models.



Hele-Shaw models : starting point

Chapter 4 of the notes. With H. Egly and R. Sentis.

Use a **cold** coupled model with one group for radiation + hydrodynamics +

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$$T = T_r \text{ and } \rho e + aT^4 = \rho C_v T + aT^4 \approx \rho C_v T.$$

The starting point of our analysis is the compressible Euler model with non linear heat flux

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0 \\ \partial_t (\rho e) + \nabla \cdot (\rho \mathbf{u} e + \mathbf{p} \mathbf{u} - \kappa n T^n \nabla T) = 0. \end{cases}$$

The Spitzer non linear coefficient is

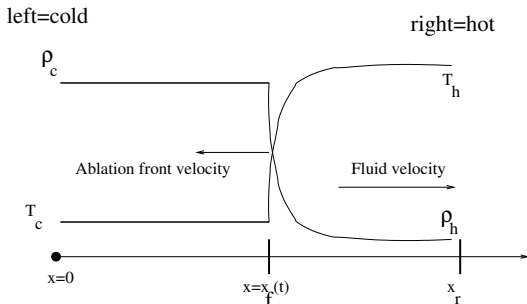
$$n \in \left[\frac{5}{2}, \frac{7}{2} \right].$$

The heat flux boundary condition on the exterior boundary Γ_r is non linear

$$\kappa n T^n \partial_n T|_{\Gamma_e} = b \text{ given.}$$

The isobar regime is

$$p = (\gamma - 1)\rho C_v T \approx C.$$



Two other important ingredients are

$$\varepsilon = \left(\frac{T_c}{T_h} \right)^n, \text{ and } \mathbf{u}(t = 0, x) = \mathbf{u}_c(t), \quad x \in \text{the cold region.}$$

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Let us define the acceleration $\mathbf{g} = \mathbf{u}'_c(t)$ and reset the velocity $\mathbf{u} \leftarrow \mathbf{u} - \mathbf{u}_c$. After rescaling one gets

$$\begin{cases} D_{t_*} \rho_* + \rho_* \nabla_* \cdot \mathbf{u}_* = 0 \\ \rho_* D_{t_*} \mathbf{u}_* + \frac{1}{M^2} \nabla_* p_* = \frac{1}{F_r} \rho_* \mathbf{g}_* \\ \frac{1}{\gamma - 1} D_{t_*} p_* + \frac{\gamma}{\gamma - 1} p_* \nabla_* \cdot \mathbf{u}_* - \nabla_* \cdot (H n T_*^n \nabla_* T_*) = 0. \end{cases}$$

where

- The Mach number $M^2 = \frac{\rho_* |u_*|^2}{p_*}$.
- The Froude number $F_r = \frac{|u_*|^2}{g_* t_*} = \frac{|u_*| t_*}{g_*}$ is a measure of the acceleration of the particles coming from the left boundary Γ_l in the cold region.
- $H = \frac{\kappa n (T_*)^{n+1}}{\rho_* |u_*|} = \frac{\kappa n (T_*)^{n+1}}{\rho_* |u_*|^3}$ measures the velocity of the particles in the hot region.



Slow Mach number expansion

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We perform an asymptotic expansion of all variables with respect to the square of the Mach number,

$$\rho_\star = \rho_\star^{(0)} + M^2 \rho_\star^{(2)} + \dots, \dots$$

One gets the quasi-isobar model

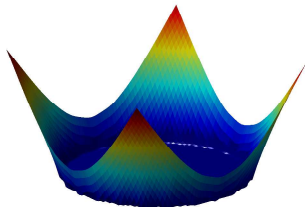
$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho \mathbf{g} \\ \nabla \cdot (\mathbf{u} - n T^n \nabla T) = 0, \\ \rho T = 1. \end{cases}$$

or also

$$\begin{cases} \partial_t T + \mathbf{u}_{\text{vort}} \cdot \nabla T - T^2 \Delta T^n = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + T \nabla p = \mathbf{g}, \\ \mathbf{u} = n T^n \nabla T + \mathbf{u}_{\text{vort}} = \mathbf{u}_{\text{therm}} + \mathbf{u}_{\text{vort}}, \\ \nabla \cdot \mathbf{u}_{\text{vort}} = 0. \end{cases}$$

Conjecture : For well prepared data, the solution of $\partial_t T - T^2 \Delta T^n = 0$ is approximated by $\max(\varepsilon, \theta)^{\frac{1}{n}}$ where $\theta \approx T^n$ is a solution of the

$$\text{Hele-Shaw equation : } \begin{cases} \Delta \theta = 0, & x \in \Omega(t), \\ \partial_n \theta = b, & x \in \Gamma_h, \\ \theta = 0, & x \in \Gamma_f(t), \\ \partial_t \Gamma_f(t) = -\nabla \theta|_{\gamma(t)}, & x \in \Gamma_f(t). \end{cases}$$



A numerical example is

Proof in 1D

Set $\Theta = T^n$. Then $\partial_t \left(\frac{1}{\Theta^n} \right) + \partial_{xx} \Theta = 0$. Progressive waves are defined by $\Theta = \Theta(x + vt)$. The generating Kull's function is the progressive wave with $v = 1$ and $\Theta(-\infty) = 1$

$$K'(x) = 1 - K^{-\frac{1}{n}}(x), \text{ normalisation } K(0) = \left(\frac{n+2}{n+1} \right)^n \approx e.$$

Set $T = \left(\varepsilon K \left(\frac{x-x_0+vt}{\varepsilon} \right) \right)^{\frac{1}{n}}$. Then $(T^n)(-\infty) = \varepsilon^{\frac{1}{n}}$, $(T^n)'(+\infty) = v$ and $\varepsilon^{\frac{1}{n}} \partial_t T - T^2 \partial_{xx} T^n = 0$.

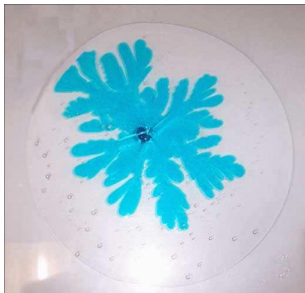
The classical Hele-Shaw problem (1898)

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HS Hele-Shaw, inventor of the Hele-Shaw cell (and the variable-pitch propeller)

$$\begin{cases} \Delta p = 0, & x \in \Omega_{\text{in}}(t) = \text{blue region}, \\ p = 0, & x \in \partial\Omega_{\text{in}}(t), \\ \partial_n x = -\nabla p, & x \in \partial\Omega_{\text{in}}(t). \end{cases}$$

The Web page of Howison (Ociam, Oxford) for some historical references about Hele-Shaw (and also with fresh science) : "Mr Hele-Shaw (inv. of the variable pitch propeller) worked on the propeller of the cruisers of her gracious majesty"

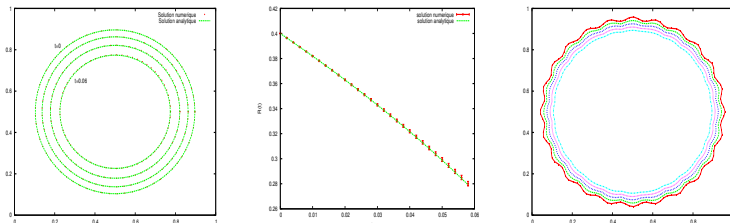
We use a Finite Element Method for the Poisson equation, and markers for the front. The markers move accordingly to the Hele-Shaw model.

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On the left $\Gamma_{\text{int}}(t)$. On the middle $t \mapsto r(t)$. On the right : The smoothing effect of the Hele-Shaw equation for convergent front is visible; In this regime, ablation fronts are stable.

The full ablative Hele-Shaw model : $\mathbf{u}_{\text{vort}} \neq 0$

The model writes

$-\Delta\theta = 0,$	$x \in \Omega - \Omega_{\text{in}},$
$\partial_n\theta = v,$	$x \in \partial\Omega = \Gamma_{\text{ext}},$
$\theta = 0,$	$x \in \partial\Omega_{\text{in}} = \Gamma_{\text{in}}(t),$
$x'(t) = \nabla\theta + \mathbf{u}_{\text{vort}},$	$x(t) \in \Gamma_{\text{in}}(t),$
$\partial_t\mathcal{V} + \nabla \cdot (\mathbf{u}_{\text{therm}}\mathcal{V}) = S_1,$	$x \in \Omega - \Omega_{\text{in}},$
$\Delta\varphi = T\mathcal{V},$	$x \in \Omega - \Omega_{\text{in}},$
$\mathbf{u}_{\text{vort}} = \nabla \wedge \varphi,$	$x \in \Omega - \Omega_{\text{in}}.$

The vorticity source is S_1 .

The thermic velocity if $\mathbf{u}_{\text{therm}} = \frac{n}{n+1} \nabla T^{n+1} = \theta^{\frac{1}{n}} \nabla\theta$.

The "vorticity" is $\mathcal{V} = \frac{\omega}{T}$.

Passive vorticity : $S_1 \neq 0$ but $u_{\text{vort}} = 0$

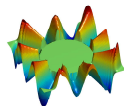
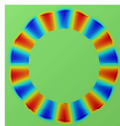
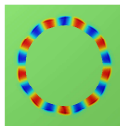
The initial data is a front Γ^{int} discretize with 100 markers and with a mode 9. We plot the vorticity.

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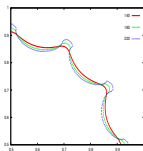
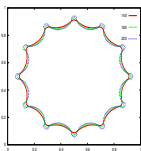
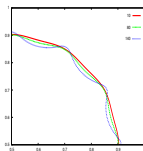
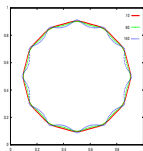
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Active vorticity : $S_1 \neq 0$ and $u_{\text{vort}} \neq 0$



In this regime, ablation fronts may be unstable.

Open problem : design a two-temperature Hele-Shaw model.