Hybrid model for the Coupling of an Asymptotic Preserving scheme with the Asymptotic Limit model

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Outline

Presentation and motivations

We are interesting in the following strongly anisotropic system :

(P)-formulation :

$$\begin{cases} -\partial_z \left(\dfrac{1}{\varepsilon} \partial_z u_\varepsilon \right) - \Delta_X u_\varepsilon + u_\varepsilon = f & , \Omega_X \times \Omega_z \\ \left(\dfrac{1}{\varepsilon} \partial_z u_\varepsilon \right) (X, \pm 1) = g_\pm (X) & , \Omega_X \\ u_\varepsilon(X, z) = 0 & , \partial\Omega_X \times \Omega_z \end{cases}$$

with $u : \Omega_X \times \Omega_z \to \mathbb{R}$, $\Omega_X = [0,1]^{d-1}$, $\Omega_z = [-1,1]$,
which degenerate to a ill-posed system at the limit $\varepsilon \to 0$.

In this work, we propose some reformulation leading to a well-posed system and the corresponding asympotic preserving scheme.

P.Degond, F.Deluzet, C. Negulescu An asymptotic preserving scheme for strongly anisotropic elliptic equation, *Preprint* 2010

Average according the anisotropy

Let introduce the splitting : $u_\varepsilon = \bar{u}_\varepsilon + u'_\varepsilon$ with $\bar{u}_\varepsilon = \frac{1}{2}\int_{\Omega_z} u_\varepsilon \mathtt{d}z$.

(AP)-formulation :

The main part satisfy :

$$-\Delta_X \bar{u}_\varepsilon + \bar{u}_\varepsilon = \bar{f} + \frac{g_+ - g_-}{2} \qquad (\bar{AP})$$

And the fluctuation :

$$\begin{cases} -\partial_z\left(\dfrac{1}{\varepsilon}\partial_z u'_\varepsilon\right) - \Delta_X u'_\varepsilon + u' = f' - \dfrac{g_+ - g_-}{2} \\ \displaystyle\int_{\Omega_z} u'_\varepsilon \mathtt{d}z = 0 \end{cases} \qquad (AP' = P - \bar{AP})$$

which is well-posed for any $\varepsilon \geq 0$.

Main probleme for applications :

Direction $\vec{e}_z$ have to know before the computationand stay fixed.

General AP model

Weak formulation of the (P)-formulation : $\forall \psi \in H_0^1(\Omega_X) \cap H^1(\Omega_z)$

$$\int_{\Omega_X \times \Omega_z} \frac{1}{\varepsilon} \partial_z u_\varepsilon \partial_z \psi \mathtt{d}X\mathtt{d}z + \int_{\Omega_X \times \Omega_z} \nabla_X u_\varepsilon \cdot \nabla_X \psi \mathtt{d}X\mathtt{d}z + \int_{\Omega_X \times \Omega_z} u_\varepsilon \psi \mathtt{d}X\mathtt{d}z$$
$$= \int_{\Omega_X \times \Omega_z} f\psi \mathtt{d}X\mathtt{d}z + \int_{\Omega_X} \left[g_+ \psi(X,1) - g_- \psi(X,-1) \right] \mathtt{d}X$$

Let introduce the auxyliary probleme :

$$\begin{cases} \varepsilon \partial_z v_\varepsilon = \partial_z u_\varepsilon & , \Omega_X \times \Omega_z \\ v_\varepsilon(X,-1) = 0 & , \Omega_X \end{cases} \qquad (P_v)$$

which is well-posed since $\varepsilon > 0$.

The new formulation could be then writen as :

$(AP4)$-formulation : $\forall \psi \in H_0^1(\Omega_X) \cap H^1(\Omega_z)$, $\forall \phi \in H_0^1(\Omega_X) \cap H_0^1(\Omega_z)$

$$\begin{cases} \displaystyle\int_{\Omega_X \times \Omega_z} \partial_z v_\varepsilon \partial_z \psi \mathtt{d}X\mathtt{d}z + \int_{\Omega_X \times \Omega_z} \nabla_X u_\varepsilon \cdot \nabla_X \psi \mathtt{d}X\mathtt{d}z + \int_{\Omega_X \times \Omega_z} u_\varepsilon \psi \mathtt{d}X\mathtt{d}z \\ \qquad\qquad\displaystyle = \int_{\Omega_X \times \Omega_z} f\psi \mathtt{d}X\mathtt{d}z + \int_{\Omega_X} \left[g_+\psi(X,1) - g_-\psi(X,-1)\right] \mathtt{d}X, \\ \displaystyle\int_{\Omega_X \times \Omega_z} \varepsilon \partial_z v_\varepsilon \phi \mathtt{d}X\mathtt{d}z = \int_{\Omega_X \times \Omega_z} \partial_z u_\varepsilon \phi \mathtt{d}X\mathtt{d}z, \\ \text{with } v_\varepsilon(-1) = 0 \end{cases}$$

which is well-posed since $\varepsilon > 0$.

What happen in $\varepsilon = 0$?
Formally, $(P_v) \Rightarrow \partial_z u_0 = 0$.
Then using ψ satisfying $\partial_z \psi = 0$, $(AP4)$ become

$$\int_{\Omega_X \times \Omega_z} \nabla_X u_0 \cdot \nabla_X \psi \mathtt{d}X + \int_{\Omega_X \times \Omega_z} u_0 \psi \mathtt{d}X = \int_{\Omega_X} \left[\int_{\Omega_z} f \mathtt{d}z + \frac{g_+ - g_-}{2}\right] \psi \mathtt{d}X.$$

Coupling of the (AP)-formulation

Let consider that ε is small enough that the approximation $\varepsilon = 0$ is acceptable, in the half-space where z is negative. We are looking for a coupling between an asymptotic preserving models and a limit model.

The main part is not depending of ε thus in the all domain we have to solve $(\bar{A}P)$.

$$\begin{cases} -\Delta_X \bar{u}_\varepsilon + \bar{u}_\varepsilon = \bar{f} + \dfrac{g_+ - g_-}{2}, \\ \qquad\qquad\qquad\qquad \bar{u}_\varepsilon = 0 \quad , \partial\Omega_X \times \Omega_z \end{cases}$$

For the perturbation term, we consider the two following problem

$$\begin{cases} -\partial_z\left(\frac{1}{\varepsilon}\partial_z u'_+\right) - \Delta_X u'_+ + u'_+ = f' - \frac{g_+ - g_-}{2} & ,\Omega_X \times [0,1] \\ \left(\frac{1}{\varepsilon}\partial_z u'_+\right)(X,1) = g_+ & ,\Omega_X \\ \left(\frac{1}{\varepsilon}\partial_z u'_+\right)(X,0) = g_\varepsilon & ,\Omega_X \\ u'_+(X,z) = 0 & \partial\Omega_X \times [0,1] \end{cases} \tag{1}$$

and

$$\begin{cases} -\partial_z\left(\frac{1}{\varepsilon}\partial_z u'_-\right) - \Delta_X u'_- + u'_- = f' - \frac{g_+ - g_-}{2} & ,\Omega_X \times [-1,0] \\ \left(\frac{1}{\varepsilon}\partial_z u'_-\right)(X,-1) = g_- & ,\Omega_X \\ \left(\frac{1}{\varepsilon}\partial_z u'_-\right)(X,0) = g_\varepsilon & ,\Omega_X \\ u'_-(X,z) = 0 & \partial\Omega_X \times [-1,0] \end{cases} \tag{2}$$

with the contraint $\int_{[-1,0]} u'_- \mathtt{d}z + \int_{[0,1]} u'_- \mathtt{d}z = 0$.

On the half-space $z < 0$, formally we have $\partial_z u'_\varepsilon = 0$, thus by continuity we have $u_\varepsilon(X, z < 0) = u_\varepsilon(X, z = 0_-) = u_\varepsilon(X, z = 0_+)$. After integration on $[-1, 0]$,

$$\lim_{\varepsilon \to 0} g_\varepsilon(X) = \Delta_X u'_\varepsilon(0) - u'_\varepsilon(0) + \frac{g_+ + g_-}{2} - \int_{-1}^{0} f' \mathtt{d}z$$

Then we write the following model :

$(AP - L)$-formulation :

$$\begin{cases} -\partial_z \left(\frac{1}{\varepsilon}\partial_z u'_\varepsilon\right) - \Delta_X u'_\varepsilon + u'_\varepsilon = f' - \dfrac{g_+ - g_-}{2} & ,\Omega_X \times [0,1] \\ u'_\varepsilon = u'_\varepsilon(0) & ,\Omega_X \times [-1,0] \\ \left(\frac{1}{\varepsilon}\partial_z u'_\varepsilon\right)(X,1) = g_+ & ,\Omega_X \\ \left(\frac{1}{\varepsilon}\partial_z u'_\varepsilon\right)(X,0) = \Delta_X u'_\varepsilon(0) - u'_\varepsilon(0) + \dfrac{g_+ + g_-}{2} - \displaystyle\int_{-1}^{0} f' \mathtt{d}z & ,\Omega_X \\ \displaystyle\int_0^1 u'_\varepsilon(X,z)\mathtt{d}X = -u'_\varepsilon(0) & ,\Omega_X \end{cases}$$

Coupling for the general (AP)-formulation

Now, we are interresting in the coupling of the second formulation.

$$\begin{cases} -\dfrac{\partial^2 v_+}{\partial z^2} - \Delta_X u + + u_+ = f & ,\Omega_X \times [0,1] \\ \varepsilon \dfrac{\partial v_+}{\partial z} = \dfrac{\partial}{\partial z} u_+ \\ v_+(X,1) = 0 \\ + \text{ bnd for } u_+ \end{cases}$$

and

$$\begin{cases} -\dfrac{\partial^2 v_-}{\partial z^2} - \Delta_X u_- + u_- = f & ,\Omega_X \times [-1,0] \\ \varepsilon \dfrac{\partial v_-}{\partial z} = \dfrac{\partial}{\partial z} u_- \\ v_-(X,-1) = 0 \\ + \text{ bnd for } u_- \end{cases}$$

Now looking at the limit in the left half domain when $\varepsilon \to 0$, formally we have $\frac{\partial u_-}{\partial z} = 0$ on $\Omega_X \times [-1, 0]$.
We multiply by a test function ψ and we integrate on $z \in [-1, 0]$, then

$$\int_{-1}^{0} \frac{\partial v_-}{\partial z} \frac{\partial \psi}{\partial z} \mathtt{d}z + \int_{-1}^{0} \Delta_X u_- \psi \mathtt{d}z + \int_{-1}^{0} u_- \psi \mathtt{d}z = \int_{-1}^{0} f \psi \mathtt{d}z + \frac{\partial v_-}{\partial z}(X, 0)\psi(X, 0) - g_- \psi(X, -1)$$

We take test function such that $\frac{\partial \psi}{\partial z} = 0$ and obtain

$$\frac{\partial v_-}{\partial z}(X, 0) = \Delta_X u_- + u_- + g_- - \int_{-1}^{0} f \mathtt{d}z$$

wich is a solution of the limit problem in $\Omega_X \times [-1, 0]$.

By continuity, we have $\forall z \in [-1,0], u_-(X,z) = u_+(X,0)$, then

$(AP4 - L)$-formulation :

$$\begin{cases} -\dfrac{\partial^2 v_+}{\partial z^2} - \Delta_X u + + u_+ = f & , \Omega_X \times [0,1] \\ \varepsilon \dfrac{\partial v_+}{\partial z} = \dfrac{\partial}{\partial z} u_+ \\ v_+(X,1) = 0 \\ \dfrac{1}{\varepsilon} \dfrac{\partial u_+}{\partial z}(X,0) = \Delta_X u_- + u_- + g_- - \displaystyle\int_{-1}^{0} f \mathtt{d} z \\ \dfrac{1}{\varepsilon} \dfrac{\partial u_+}{\partial z}(X,1) = g_+ \end{cases}$$

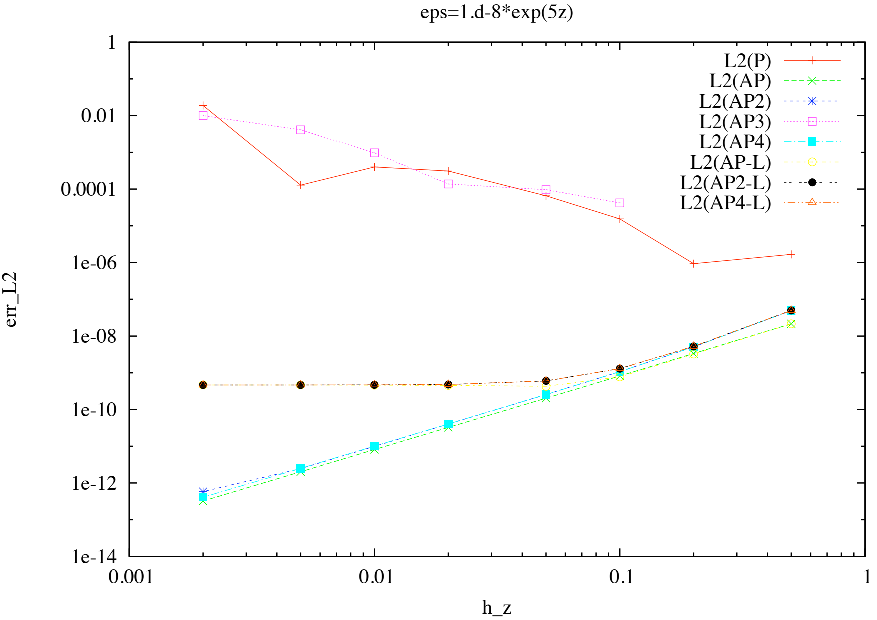
Simulations in 1D

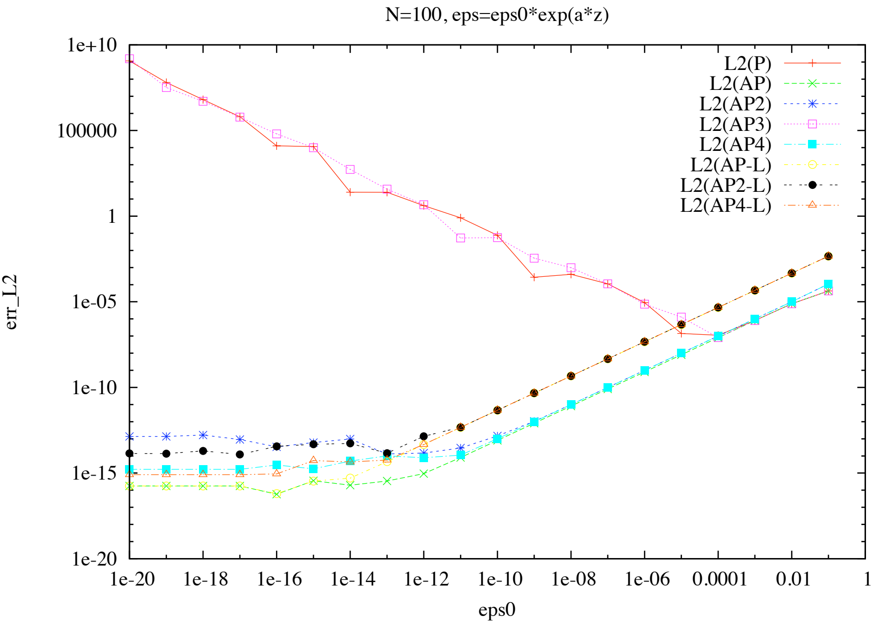
Model characteristicsfor a $\mathbb{P}_1$-finite element method

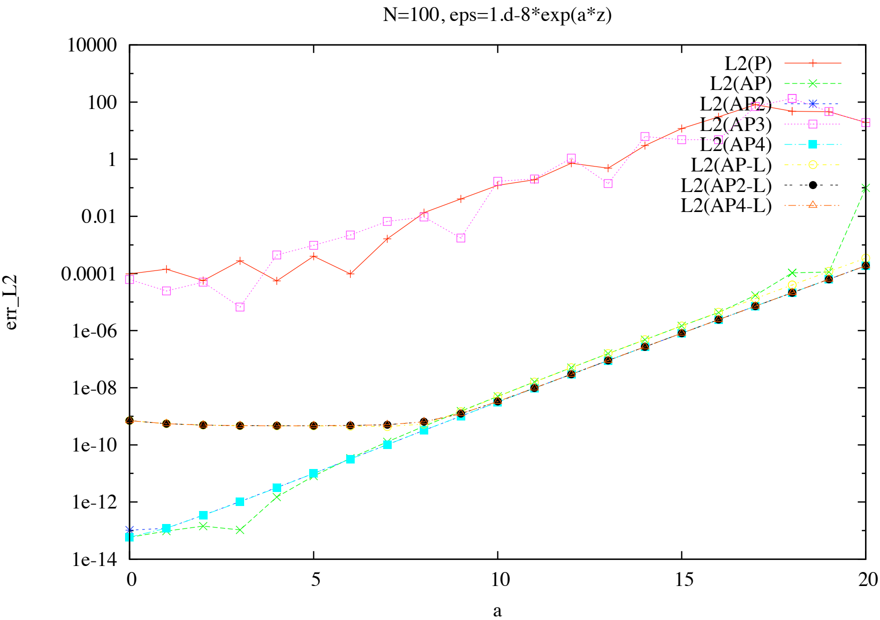
Model	Matrix	size	non zero coeff	cond
(P)	$\begin{pmatrix} \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \end{pmatrix}$	2N+1	6N+1	$\frac{1}{\varepsilon h_z^2}$
(AP)	$\begin{pmatrix} \star & \star & 0 & 0 & 0 & \star \\ \star & \star & \star & 0 & 0 & \star \\ 0 & \star & \star & \star & 0 & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ \star & \star & \star & \star & \star & 0 \end{pmatrix}$	2N+2	10N+3	$\frac{1}{h_z^2}$
$(AP-L)$		N+2	5N+3	$\frac{1}{h_z^2}$
$(AP2)$	$\begin{pmatrix} \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 \\ 0 & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star \end{pmatrix}$	4N+2	24N	$\frac{1}{h_z^2}$
$(AP2-L)$		2N+2	12N	$\frac{1}{h_z^2}$
$(AP3)$		4N+2	24N+2	$\frac{1}{\varepsilon h_z^2}$
$(AP4)$		4N+2	20N+2	$\frac{1}{h_z^2}$
$(AP4-L)$		2N+2	10N+2	$\frac{1}{h_z^2}$

Cas test :

$u(z) = 10 + \varepsilon \cos(\pi z)$, $\qquad \varepsilon(z) = \varepsilon_0 e^{az}$


eps=1.d-8*exp(5z)
err_L2
h_z
1
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
0.001
0.01
0.1
1
L2(P)
L2(AP)
L2(AP2)
L2(AP3)
L2(AP4)
L2(AP-L)
L2(AP2-L)
L2(AP4-L)


N=100, eps=eps0*exp(a*z)
L2(P)
L2(AP)
L2(AP2)
L2(AP3)
L2(AP4)
L2(AP-L)
L2(AP2-L)
L2(AP4-L)
err_L2
1e+10
100000
1
1e-05
1e-10
1e-15
1e-20
1e-20
1e-18
1e-16
1e-14
1e-12
1e-10
1e-08
1e-06
0.0001
0.01
1
eps0


N=100, eps=1.d-8*exp(a*z)
10000
100
1
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
err_L2
0
5
10
15
20
a
L2(P)
L2(AP)
L2(AP2)
L2(AP3)
L2(AP4)
L2(AP-L)
L2(AP2-L)
L2(AP4-L)

Thanks for your attention!