HYDROMELC
Electrostatic charge phenomena : hydrodynamic model and interface conditions.

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Outline

Motivation

- In magnetosphere, satellites are surrounded by a plasma : interaction between charged particles of the plasma (ion and electrons) and the spacecraft.
- Ions and electrons of the plasma modify the electrostatic charge of the external surfaces of the spacecraft.
- This leads to charging phenomena. An excessive charging induces the formation of electric arching which can destroy the experimental devices.

$\Longrightarrow$ For the numerical simulation of the spacecraft charging, we need a complete description of the boundary conditions at the interface between the plasma and the spacecraft.

Motivation

Kinetic description of a plasma

- $F_e = F_e(t, x, v)$ and F_i the densities functions of electrons and ions respectively.
- Boltzmann-Poisson system (for $\beta \in \{i, e\}$) :

$$\frac{\partial F_\beta}{\partial t} + v \cdot \nabla_x F_\beta - \frac{q_\beta}{m_\beta} \nabla_x \Phi_\beta \cdot \nabla_v F_\beta = Q_\beta(F_i, F_e),$$
$$-div_x(\epsilon_0 \nabla_x V_\beta) = q_i n_i + q_e n_e,$$
$$n_\beta = \int F_\beta \, dv.$$

- Boundary conditions on F_e and F_i at the boundary Γ of the satellite :

$$\gamma^{inc} F_\beta = \mathcal{R}(\gamma^{out} F_\beta) + S \text{ for } v \cdot \eta(x) < 0, \quad x \in \Gamma$$

$+$ Boundary conditions on V_β

- Initial conditions on V_β and F_β.

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- Hydrodynamic limit of Boltzmann-Poisson equations : Euler equation on $(n_\beta, u_\beta, \theta_\beta)$ defined by

$$\begin{cases} n_\beta = \displaystyle\int F_\beta dv \\ n_\beta u_\beta = \displaystyle\int v F_\beta(v) dv \\ n_\beta u_\beta^2 + N n_\beta \theta_\beta = \displaystyle\int |v|^2 F_\beta(v) dv. \end{cases}$$

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- What are the boundary conditions to impose for n_β, u_β, and θ_β ?
- Difficulties :
  - imposing only the incoming flux,
  - taking into account the Knudsen layer.

Objective of the study

Hypothesis

- N=1, $\Omega = [-\omega, \omega]$
- Only one specie (ions or electrons)
- BGK operator :

$$Q(F) = M[F] - F, \quad \text{with} \quad M[F](v) = M_{\rho,u,\theta} = \frac{\rho}{\sqrt{2\pi\theta}} \exp\left(-\frac{|v-u|^2}{2\theta}\right)$$

BGK-Poisson model

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F - q\partial_x \Phi \partial_v F = \frac{1}{\tau}\left(M[F] - F\right)$$
$$-\partial_x^2 V = q(\rho - 1), \quad q \in \{-1, 1\}$$
$$V(-\omega) = V^L, \quad V(\omega) = V^R.$$

Boundary conditions

$$\begin{aligned}
\gamma^{inc}F(t,-\omega,v) &= \mathcal{R}(\gamma^{out}F(t,-\omega,\cdot))(v) + \phi^{data,L}(t,v) \text{ for } v>0 \\
\gamma^{inc}F(t,\omega,v) &= \mathcal{R}(\gamma^{out}F(t,\omega,\cdot))(v) + \phi^{data,R}(t,v) \text{ for } v<0
\end{aligned}$$

- Diffuse reflexion (for the boundary $x=-\omega$)

$$\mathcal{R}(\gamma^{out}F(t,-\omega,\cdot))(v) = \alpha\frac{M_w(v)}{Z_w}\int_{v'<0}|v'|\gamma^{out}F(t,-\omega,v')\,dv' \text{ for } v>0,$$

where $\alpha\in[0,1]$ and

$$M_w(v) = \frac{\rho_w}{\sqrt{2\pi\theta_w}}e^{-|v|^2/(2\theta_w)}, \quad Z_w = \int_{v>0} vM_w(v)\,dv.$$

- Specular reflexion (for the boundary $x=-\omega$)

$$\mathcal{R}(\gamma^{out}F(t,-\omega,\cdot))(v) = \alpha\gamma^{out}F(t,-\omega,-v) \text{ for } v>0. \tag{1}$$

References

We follow the approach described in :

F. Coron, F. Golse, C. Sulem, *A classification of well-posed kinetic layer problems*, Comm. Pure Appl. Math., **41** (4), 409–435 (1988).

C. Bardos, F. Golse ; Y. Sone, *Half-space problems for the Boltzmann equation : a survey*, Journal of Statistical Physics, **124**, (2)-(4), (2006).

C. Besse, S. Borghol, T. Goudon, I. Lacroix-Violet, J.-P. Dudon, *Hydrodynamic regimes, Knudsen layer, numerical schemes : definition of boundary fluxes*, submitted.

Outline

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Linearized BGK equation

We set $F = M_*(1+\delta f)$, with $M_* := M_{(\rho_*, u_*, \theta_*)}$, and we obtain

$$\partial_t f + v\partial_x f = \frac{1}{\tau} L_*(f),$$

with $L_*(f) = \Pi f - f$:

$$\Pi f = \frac{\widetilde{\rho}}{\rho_*} + \frac{v-u_*}{\theta_*}\widetilde{u} + \left(\frac{|v-u_*|^2}{\theta_*} - 1\right)\frac{\widetilde{\theta}}{2\theta_*} =: m_{(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})},$$

where $(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})$ are such that

$$\begin{pmatrix} \widetilde{\rho} \\ \widetilde{\rho} u_* + \rho_* \widetilde{u} \\ \widetilde{\rho}(u_*^2 + \theta_*) + 2\rho_* u_* \widetilde{u} + \rho_* \widetilde{\theta} \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(\cdot, \cdot, v) M_*(v)\, dv.$$

We have $\mathrm{Ker}(L_*) = \mathrm{Span}\ \{1, v, |v|^2\}$.

Linearized Euler system

- Substituting f by the infinitesimal Maxwellian $m_{(\widetilde{\rho},\widetilde{u},\widetilde{\theta})}$ leads to the linearized Euler system :

$$\partial_t \begin{pmatrix} \widetilde{\rho} \\ \widetilde{u} \\ \widetilde{\theta} \end{pmatrix} + \begin{pmatrix} u_* & \rho_* & 0 \\ \frac{\theta_*}{\rho_*} & u_* & 1 \\ 0 & 2\theta_* & u_* \end{pmatrix} \partial_x \begin{pmatrix} \widetilde{\rho} \\ \widetilde{u} \\ \widetilde{\theta} \end{pmatrix} = 0. \tag{2}$$

- Eigenvalues : $\{u_* - \sqrt{3\theta_*}, u_*, u_* + \sqrt{3\theta_*}\}$.
- Entropy flux :

$$\mathbf{Q} : \widetilde{U} = (\widetilde{\rho}, \widetilde{u}, \widetilde{\theta}) \mapsto \int_{\mathbb{R}} v |m_{(\widetilde{\rho},\widetilde{u},\widetilde{\theta})}|^2 \, M_* \, dv.$$

- We can split $Ker(L_*)$ according to the sign of the quadratic form $\mathbf{Q}$

$$Ker(L_*) = \Lambda^+ \oplus \Lambda^- \oplus \Lambda^0.$$

Half space problem

- Ansatz for f :

$$f(t,x,v) = m_{(\widetilde{\rho},\widetilde{u},\widetilde{\theta})} + G^L\left(t, \frac{x+\omega}{\tau}\right) + G^R\left(t, \frac{x-\omega}{\tau}\right) + o(\tau).$$

- G^R and G^L stand for boundary layers and are solution of the half space problem

$$\begin{cases} v\partial_z G = L_* G, & z>0,\ v\in\mathbb{R} \\ G(0,v) = \Upsilon^{data} & \text{for} \quad v>0. \end{cases} \tag{3}$$

- *Theorem* : There exists a unique $G \in L^\infty(0,\infty; L^2(\mathbb{R}, M_*(v)\,dv))$ solution of (3) and a linear mapping (generalized Chandrasekhar functional)

$$\begin{array}{llll} \mathcal{C}_* : & L^2(\mathbb{R}, (1+|v|)M_*(v)\,dv) & \to & \Lambda^+ \oplus \Lambda^0 \\ & \Upsilon^{data} & \mapsto & m_\infty, \end{array}$$

where $m_\infty = \lim_{z\to\infty} G(z)$.

- For $x=-\omega$: G^L is defined by (3) with $\Upsilon^{data} = \gamma^{inc} f(\cdot,-\omega,\cdot) - m_{(\widetilde{\rho},\widetilde{u},\widetilde{\theta})}$.

Flux at the boundary (example of $x = -\omega$)

- Decomposition of $m_{(\widetilde{\rho},\widetilde{u},\widetilde{\theta})}(t,-\omega,v) \in Ker(L_*) = \Lambda^+ \oplus \Lambda^0 \oplus \Lambda^-$:

$$m_{(\widetilde{\rho},\widetilde{u},\widetilde{\theta})} = m_+ + m_-,$$

with $m_+ = \lim_{z\to\infty} G^L(z)(= m_\infty)$.

- $m_- \in \Lambda^-$: outcoming flux
$\rightarrow$ given by the fluid.

$$m_-(v) = \sum_{k\in I^-} \alpha_k \xi_k, \quad \text{where } (\xi_k)_{k\in I^-} \text{ is a basis of } \Lambda^-.$$

- $m_+ \in \Lambda^+ \oplus \Lambda^0$: incoming flux
$\rightarrow$ to be imposed as a boundary condition on the Euler system.

- Determination of m_+ : by integration of (3), we get

$$\frac{d}{dz}\int_{\mathbb{R}} v \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} G(z,v)M_*(v)dv = 0$$

- Maxwell approximation : $\gamma^{out} G(0,v) = G(\infty, v)$.
- We obtain :

$$\int_{v>0} v \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \left[\gamma^{inc}(f) - m_-\right] M_* dv = \int_{v>0} v \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} m_+ M_* dv$$

with

$$\gamma^{inc}(f)(v) = \frac{1}{M_*(v)}\mathcal{R}(\gamma^{out}(fM_*)(t,-\omega,\cdot))(v) + \Psi^{data,L}(t,v) \text{ for } v > 0.$$

Numerical examples

We compare

- $(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})$ given by

$$\begin{pmatrix} \widetilde{\rho} \\ \widetilde{\rho} u_* + \rho_* \widetilde{u} \\ \widetilde{\rho}(u_*^2 + \theta_*) + 2\rho_* u_* \widetilde{u} + \rho_* \widetilde{\theta} \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f M_* \, dv.$$

  where f is the solution of the BGK equation :

$$\partial_t f + v \partial_x f = \frac{1}{\tau} L_*(f) \quad (+\text{B.C}),$$

- and $(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})$ solution of the linearized Euler system :

$$\partial_t \begin{pmatrix} \widetilde{\rho} \\ \widetilde{u} \\ \widetilde{\theta} \end{pmatrix} + \begin{pmatrix} u_* & \rho_* & 0 \\ \frac{\theta_*}{\rho_*} & u_* & 1 \\ 0 & 2\theta_* & u_* \end{pmatrix} \partial_x \begin{pmatrix} \widetilde{\rho} \\ \widetilde{u} \\ \widetilde{\theta} \end{pmatrix} = 0 \quad (+\text{B.C}).$$

Example 1

- $(\rho_*, u_*, \theta_*) = (1, -0.1, 1)$
  $\rightarrow$ signature of Q : $(1, 2)$
- specular reflexion
- $\alpha = 0.1$

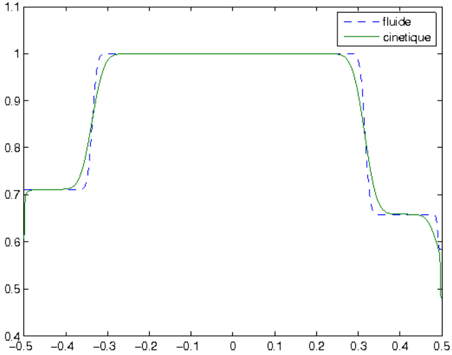


FIG.: Density at t=0.1 s

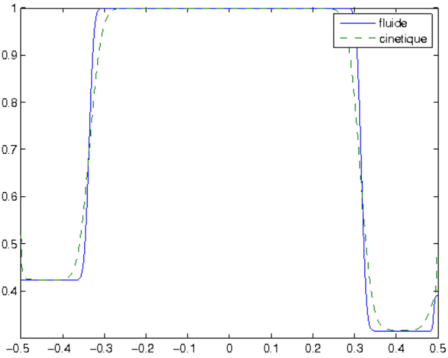


Fig.: Temperature at t=0.1 s

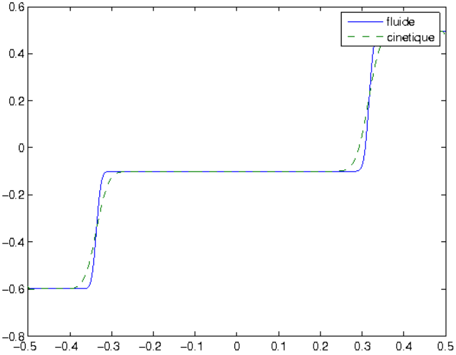


Fig.: Macroscopic velocity at t=0.1 s

Example 2

- $(\rho_*, u_*, \theta_*) = (1, 0, 1)$
  $\rightarrow$ signature of Q : $(1, 1)$
- diffuse reflexion
- $(\rho_w, \theta_w) = (1, 1)$
- $\alpha = 0.8$

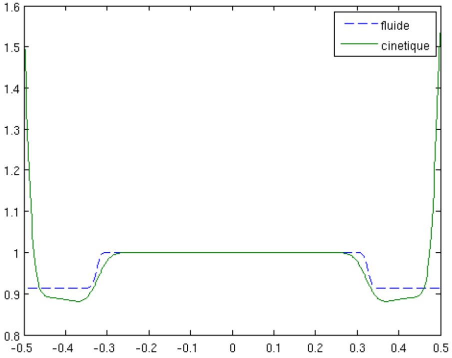
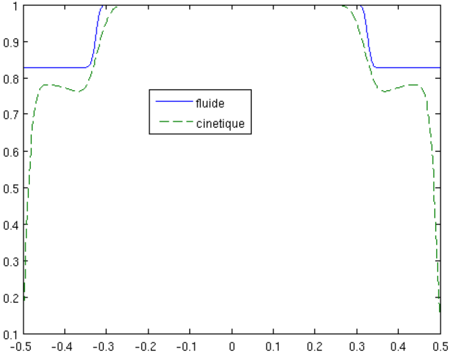


FIG.: Density at t=0.1 s



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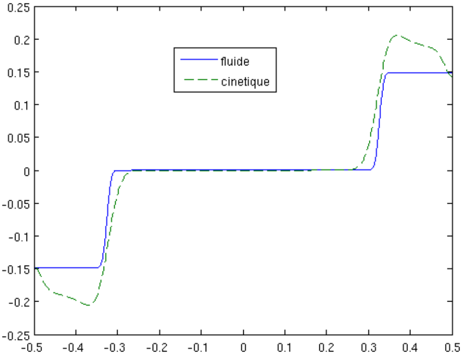


Fig.: Macroscopic velocity at t=0.1 s

Outline

Linearization

- We set for equilibrium state : $\mathcal{M}_*(x,v) = M_{(\rho_*,0,\theta_*)}(v)\dfrac{e^{-qV_*(x)/\theta_*}}{\int_{-\omega}^{\omega} e^{-qV_*(y)/\theta_*}\,dy}$.

  where $V_* \in C^2(-\omega,\omega)$ is the (unique) solution of the problem

$$\begin{cases} -\partial_x^2 V_* = q\left(\rho_* \dfrac{e^{-qV_*(x)/\theta_*}}{\int_{-\omega}^{\omega} e^{-qV_*(y)/\theta_*}\,dy} - 1\right), \\ V_*(-\omega) = V^L, \quad V_*(\omega) = V^R. \end{cases} \tag{4}$$

- The linearization $F = \mathcal{M}_*(1+\delta f)$ and $V = V_* + \delta\widetilde{V}$ lead to the linearized Boltzmann-Poisson problem :

$$\begin{cases} \partial_t f + v\partial_x f - q\partial_x V_* \partial_v f + q\dfrac{v}{\theta_*}\partial_x \widetilde{V} = \dfrac{1}{\tau} L_* f, \\ -\partial_x^2 \widetilde{V} = q\displaystyle\int_{\mathbb{R}} f\mathcal{M}_*\,dv, \quad \widetilde{V}(-\omega) = \widetilde{V}(\omega) = 0. \end{cases}$$

Linearized Euler-Poisson system

Substituting f by the infinitesimal Maxwellian $m_{(\widetilde{\rho},\widetilde{u},\widetilde{\theta})}$ leads to :

$$\partial_t \begin{pmatrix} \widetilde{\rho} \\ \widetilde{u} \\ \widetilde{\theta} \end{pmatrix} + \begin{pmatrix} 0 & \rho_* & 0 \\ \frac{\theta_*}{\rho_*} & 0 & 1 \\ 0 & 2\theta_* & 0 \end{pmatrix} \partial_x \begin{pmatrix} \widetilde{\rho} \\ \widetilde{u} \\ \widetilde{\theta} \end{pmatrix} = q\partial_x V_* \begin{pmatrix} \frac{\rho_* \widetilde{u}}{\theta_*} \\ \frac{\widetilde{\theta}}{\theta_*} \\ 0 \end{pmatrix} - q\partial_x \widetilde{V} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} .$$

with

$$-\partial_x^2 \widetilde{V} = q\,\widetilde{\rho} \frac{e^{-qV_*(x)/\theta_*}}{\int_{-\omega}^{\omega} e^{-qV_*(y)/\theta_*}\,dy}, \qquad \widetilde{V}(-\omega) = \widetilde{V}(\omega) = 0.$$

Numerical examples

Example 3

- $(\rho_*, u_*, \theta_*) = (1, 0, 0.5)$
- Source term at the left boundary : $\phi^{data,L} = m_{(\rho_w^L, 0, \theta_w^L)}$, with

  $(\rho_w^L, \theta_w^L) = (2/1.2, 1.2/2)$
- Boundary condition at the right side : $\phi^{data,L} = m_{(\rho_*, 0, \theta_*)}$
- $q = -1$
- $V^R = 1$, $V^L = 1$.

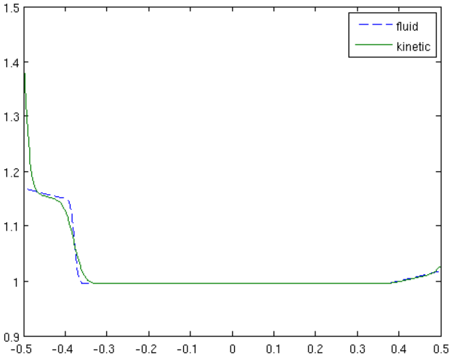


FIG.: Density at t=0.1 s

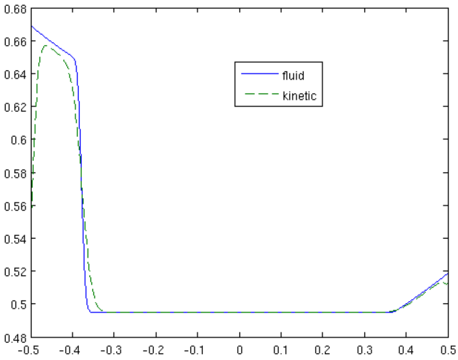


FIG.: Temperature at t=0.1 s

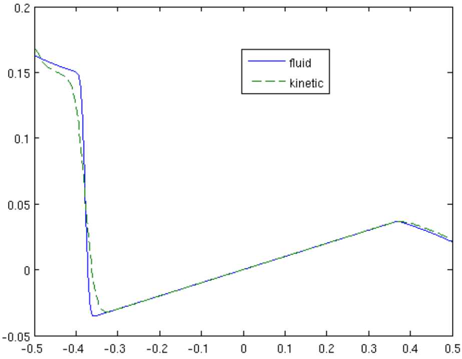


FIG.: Macroscopic velocity at t=0.1 s

Example 4

- $(\rho_*, u_*, \theta_*) = (1, 0, 0.5)$
- specular reflexion
- $\alpha = 0.4$
- $q = 1$
- $V^R = 0$, $V^L = 1$.

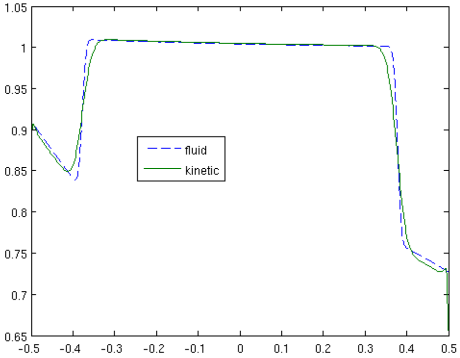
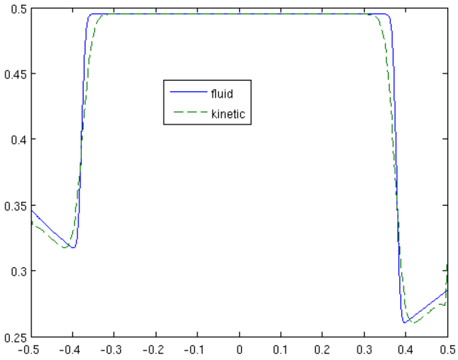


FIG.: Density at t=0.1 s



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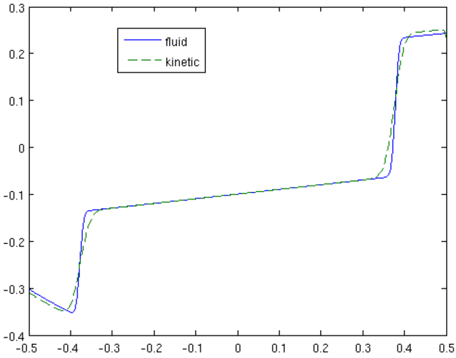

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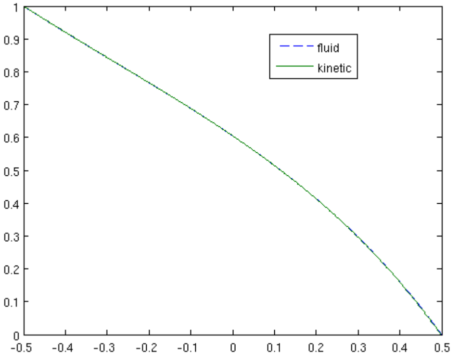

FIG.: Potential V at t=0.1 s

Outline

Conclusion

- Comparison between linearized BGK-Poisson equation and linearized Euler system.
- In most of situation, comparisons are quite good !
- Successful addition of the Poisson potential.

Work in progress

- Improvment of the case with diffuse reflexion and α close to 1.
- Implementation of the full Euler-Poisson system and comparison with the BGK-Poisson equation.

Perspectives

- The full ions-electrons system.