Nonlinear Temperature Evolution Simulation for tokamak plasma

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Context

Problem: Description of the transport of magnetically confined fusion plasma in the edge region (SOL) of a tokamak.
TOKAM3D: A model to study the instabilities occuring in the plasma edge region.

$$\begin{cases} \partial_t n_\alpha + \nabla \cdot (n_\alpha u_\alpha) = 0, \\ m_\alpha n_\alpha \left[\partial_t u_\alpha + (u_\alpha \cdot \nabla) u_\alpha\right] = -\nabla p_\alpha + n_\alpha e_\alpha (E + u_\alpha B) - \nabla \cdot \Pi_\alpha + R_\alpha, \\ \frac{3}{2} n_\alpha \left[\partial_t T_\alpha + (u_\alpha \cdot \nabla) T_\alpha\right] + p_\alpha \nabla \cdot u_\alpha = -\nabla \cdot q_\alpha - \Pi_{\alpha_{xy}} \partial_y u_{\alpha_x} + Q_\alpha, \end{cases}$$

where $\alpha = e$ for electrons and $\alpha = i$ for ions, the pressure p_α, the pressure tensor Π_α and the energy flux q_α are specified.

Simplified model equation

The system, that we are interested in, is composed of the evolution equation

$$\partial_t T_\alpha - \partial_s(K_{\|,\alpha} T_\alpha^{5/2} \partial_s T_\alpha) - \partial_r(K_{\perp,\alpha} \partial_r T_\alpha) = \pm\beta(T_i - T_e), \text{ for } (s,r) \in \Omega,$$

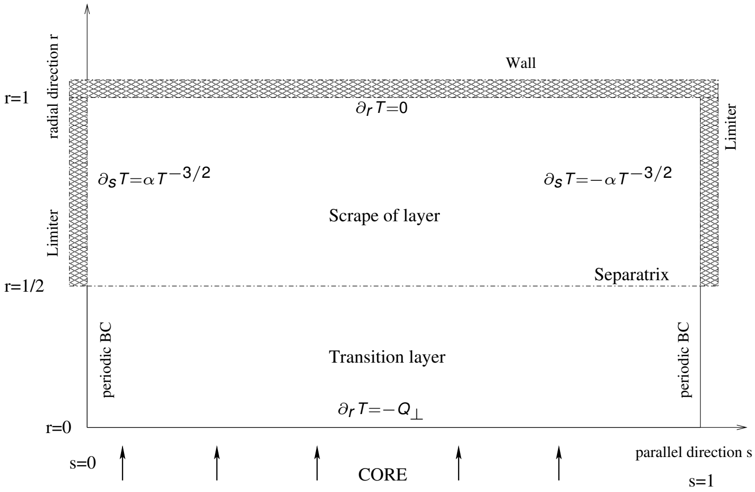
completed with the boundary conditions

$$\begin{cases} \partial_r T_\alpha = -Q_{\perp,\alpha}, & \text{for } s \in (0,1), r = 0, \\ \partial_r T_\alpha = 0, & \text{for } s \in (0,1), r = 1, \\ K_{\|,\alpha} T_\alpha^{5/2} \partial_s T_\alpha = \gamma_\alpha T_\alpha, & \text{for } s = 0, r \in (1/2, 1), \\ K_{\|,\alpha} T_\alpha^{5/2} \partial_s T_\alpha = -\gamma_\alpha T_\alpha, & \text{for } s = 1, r \in (1/2, 1), \\ \text{periodic}, & \text{for } s = 0; 1, r \in (0, 1/2), \end{cases}$$

and the initial condition

$$T_\alpha(0) = T_\alpha^0.$$

2D domain



radial direction r
Wall
r=1
$\partial_r T = 0$
Limiter
$\partial_s T = \alpha T^{-3/2}$
$\partial_s T = -\alpha T^{-3/2}$
Scrape of layer
Limiter
Separatrix
r=1/2
periodic BC
Transition layer
periodic BC
$\partial_r T = -Q_\perp$
r=0
parallel direction s
s=0
CORE
s=1

Outline

1. 1D nonlinear problem
2. 2D complete problem
3. Coupling problem

Description of 1D nonlinear problem

The 1D equation is given by

$$\partial_t T - \partial_s (K_{\|} T^{5/2} \partial_s T) = 0, \text{ for } s \in (0,1),$$

with boundary condtions as follows

$$\begin{cases} K_{\|} T^{5/2} \partial_s T = \gamma T, & \text{for } s = 0, \\ K_{\|} T^{5/2} \partial_s = -\gamma T, & \text{for } s = 1, \end{cases}$$

and the initial condition

$$T(0) = T^0.$$

We use a partition in axis s as follows

$$0 = s_{-1/2} < s_{1/2} < \cdots < s_{i-1/2} < \cdots < s_{N_s - 1/2} = 1,$$

and the unknowns $(T_i)_{i=0}^{Ns-1}$ are the approximations of temperature at the cell mid-points.

Explicit scheme I

The discretization of 1D equation can be written as

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{F_{i+1/2} - F_{i-1/2}}{\Delta s}, \; i = 0, \dots, N_s - 1,$$

where the heat fluxes are discretized as

$$\begin{aligned} F_{i-1/2} &= K_{\parallel}(T_{i-1/2}^n)^{5/2}(\partial_s T^n)_{i-1/2} \\ &= K_{\parallel}\left[\frac{T_i^n + T_{i-1}^n}{2}\right]^{5/2}\frac{T_i^n - T_{i-1}^n}{\Delta s}, \; i = 1, \dots, N_s - 1, \\ F_{i+1/2} &= K_{\parallel}(T_{i+1/2}^n)^{5/2}(\partial_s T^n)_{i+1/2} \\ &= K_{\parallel}\left[\frac{T_{i+1}^n + T_i^n}{2}\right]^{5/2}\frac{T_{i+1}^n - T_i^n}{\Delta s}, \; i = 0, \dots, N_s - 2. \end{aligned}$$

Explicit scheme II

This scheme is completed with the boundary fluxes $F_{-1/2}$ resp. $F_{Ns-1/2}$ as

$$\begin{aligned} F_{-1/2} &= K_{\|}(T^n_{-1/2})^{5/2}(\partial_s T^n)_{-1/2} = \gamma T^n_{-1/2} \approx \gamma T^n_0, \\ F_{N_s-1/2} &= K_{\|}(T^n_{i+1/2})^{5/2}(\partial_s T^n)_{i+1/2} = \gamma T^n_{Ns-1/2} \approx \gamma T^n_{Ns-1}. \end{aligned}$$

For the stability of explicit scheme, we require that

$$\Delta t^{n+1} \leq \min\left(\Delta t^n, \frac{\Delta s^2}{2\max\left(K_{\|}\|T^n\|_\infty^{5/2}, \gamma\Delta s\right)}\right).$$

☞ This scheme is costly when mesh is vary fine.

Implicit scheme

The numerical fluxes are given as follows

$$
\begin{aligned}
F_{i-1/2} &= K_{\parallel}(T^n_{i-1/2})^{5/2}(\partial_s T^{n+1})_{i-1/2} \\
&= K_{\parallel}\left[\frac{T^n_i + T^n_{i-1}}{2}\right]^{5/2}\frac{T^{n+1}_i - T^{n+1}_{i-1}}{\Delta s},\ i = 1, \ldots, N_s - 1, \\
F_{i+1/2} &= K_{\parallel}(T^n_{i+1/2})^{5/2}(\partial_s T^{n+1})_{i+1/2} \\
&= K_{\parallel}\left[\frac{T^n_{i+1} + T^n_{i}}{2}\right]^{5/2}\frac{T^{n+1}_{i+1} - T^{n+1}_{i}}{\Delta s},\ i = 0, \ldots, N_s - 2, \\
F_{-1/2} &= \gamma T^{n+1}_0, \\
F_{N_s-1/2} &= \gamma T^{n+1}_{Ns-1}.
\end{aligned}
$$

- This scheme is unconditionally stable.
- But each iteration is costly.

Filbet-Jin scheme I

Main idea

Replace the nonlinear term in matrix of linear system by a constant ν.

The semi-discretization of Filbet-Jin scheme

$$\frac{T^{n+1} - T^n}{\Delta t} - \partial_s \left(\nu \partial_s T^{n+1} \right) = \partial_s \left(\left(K_{\parallel} \left(T^n \right)^{5/2} - \nu \right) \partial_s T^n \right), \qquad (1)$$

with boundary conditions

$$\nu \partial_s T^{n+1} + \gamma T^{n+1} = \left(K_{\parallel} \left(T^n \right)^{5/2} - \nu \right) \partial_s T^n, \text{ for } s = 0,$$
$$\nu \partial_s T^{n+1} - \gamma T^{n+1} = \left(K_{\parallel} \left(T^n \right)^{5/2} - \nu \right) \partial_s T^n, \text{ for } s = 1.$$

Filbet-Jin scheme II

Then we multiply (1) by T^{n+1} and integrate in interval $(0,1)$, we have

$$\frac{1}{2}\int_0^1 \left(T^{n+1}\right)^2 ds - \frac{1}{2}\int_0^1 (T^n)^2 ds \le \int_0^1 \left(\left(T^{n+1}\right)^2 - T^{n+1}T^n\right) ds$$
$$\le \Delta t \int_0^1 \left(\left(\nu - K_{\|}(T^n)^{5/2}\right)\partial_s T^n \partial_s T^{n+1} - \nu\left(\partial_s T^{n+1}\right)^2\right) ds$$
$$-\Delta t\gamma\left(\left(T_0^{n+1}\right)^2 + \left(T_{N_s-1}^{n+1}\right)^2\right).$$

If we assume $0 \le \nu - K_{\|}(T^n)^{5/2} \le \nu$, by applying the Young's inequality, we obtain

$$\left(\nu - K_{\|}(T^n)^{5/2}\right)\partial_s T^n \partial_s T^{n+1} - \nu\left|\partial_s T^{n+1}\right|^2$$
$$\le \frac{\varepsilon}{2}\left|\partial_s T^n\right|^2 + \frac{\nu^2}{2\varepsilon}\left|\partial_s T^{n+1}\right|^2 - \nu\left|\partial_s T^{n+1}\right|^2.$$

Filbet-Jin scheme III

If $\varepsilon = \nu$, we have

$$
\begin{aligned}
& \int_0^1 \left(T^{n+1}\right)^2 ds + \Delta t \nu \int_0^1 \left|\partial_s T^{n+1}\right|^2 ds \\
\leq & \int_0^1 (T^n)^2 ds + \Delta t \nu \int_0^1 |\partial_s T^n|^2 ds \\
\vdots & \\
\leq & \int_0^1 \left(T^0\right)^2 ds + \Delta t \nu \int_0^1 \left|\partial_s T^0\right|^2 ds.
\end{aligned}
$$

Hence, the Filbet-Jin scheme is stable when $K_{\|} \| T^n \|_{\infty}^{5/2} \leq \nu$.

Numerical results I

The parameters:

$$\gamma = 2, K_{\|} = 1.$$

The initial temperature:

$$T^0 = 5.$$

Total time is 2.
Computation time of explicit scheme

N_s	50	100
Time (s)	36.42	249.07

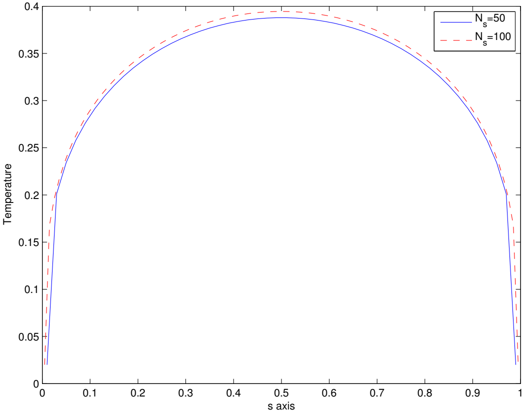


Figure: Evolution of temperature

Numerical results II

Computation time of different schemes:

Δt		0.1	0.01	0.001	0.0001
Implicit scheme	$N_s = 50$	0.01	0.07	0.49	4.99
	$N_s = 100$	0.02	0.12	1.23	12.31
F-J scheme	$N_s = 50$	0.01	0.02	0.20	1.47
	$N_s = 100$	0.03	0.06	0.36	3.32

Error estimation: $\| \cdot \| = \| T_{\text{explicit}} - T \|_2 / \| T_{\text{explicit}} \|_2$

Δt		0.1	0.01	0.001	0.0001
Implicit scheme	$N_s = 50$	0.0134	0.0011	1.08e-4	8.78e-6
	$N_s = 100$	0.0124	0.0010	1.02e-4	9.64e-6
F-J scheme	$N_s = 50$	0.0318	0.0015	1.36e-4	9.52e-6
	$N_s = 100$	0.0354	0.0016	1.08e-4	1.07e-5

Description of 2D complete problem

The 2D equation is given by

$$\partial_t T - \partial_s(K_{\|} T^{5/2} \partial_s T) - \partial_r(K_{\perp} \partial_r T) = 0, \text{ for } (s, r) \in \Omega,$$

with boundary condtions as follows

$$\begin{cases} \partial_r T = -Q_{\perp}, & \text{for } s \in (0,1), r = 0, \\ \partial_r T = 0, & \text{for } s \in (0,1), r = 1, \\ K_{\|} T^{5/2} \partial_s T = \gamma T, & \text{for } s = 0, r \in (1/2, 1), \\ K_{\|} T^{5/2} \partial_s T = -\gamma T, & \text{for } s = 1, r \in (1/2, 1), \\ \text{periodic}, & \text{for } s = 0; 1, r \in (0, 1/2). \end{cases}$$

Lie splitting decomposition

We update first in direction s

$$\mathcal{T} \quad \begin{cases} \frac{T^* - T^n}{\Delta t} - \partial_s (K_{\parallel} T^{5/2} \partial_s T) = 0, & \text{for } (s, r) \in \Omega_T, \\ \text{periodic}, & \text{for } s = 0; 1, r \in (0, 1/2), \end{cases}$$

$$\mathcal{S} \quad \begin{cases} \frac{T^* - T^n}{\Delta t} - \partial_s (K_{\parallel} T^{5/2} \partial_s T) = 0, & \text{for } (s, r) \in \Omega_S, \\ K_{\parallel} T^{5/2} \partial_s T = \gamma T, & \text{for } s = 0, r \in (1/2, 1), \\ K_{\parallel} T^{5/2} \partial_s = -\gamma T, & \text{for } s = 1, r \in (1/2, 1), \end{cases}$$

Then we update in direction r

$$\mathcal{R} \quad \begin{cases} \frac{T^{n+1} - T^*}{\Delta t} - \partial_r (K_{\perp} \partial_r T) = 0, & \text{for } (s, r) \in \Omega, \\ \partial_r T = -Q_{\perp}, & \text{for } s \in (0, 1), r = 0, \\ \partial_r T = 0, & \text{for } s \in (0, 1), r = 1. \end{cases}$$

Filbet-Jin discretization

Here, we focus on the equation $\mathcal{S}$. The Filbet-Jin form of equation $\mathcal{S}$ is

$$\frac{T^* - T^n}{\Delta t} - \partial_s \left(\nu_j \partial_s T^* \right) = \partial_s \left(\left(K_{\parallel} \left(T^n \right)^{5/2} - \nu_j \right) \partial_s T^n \right), \quad j = 0, \ldots, N_r - 1,$$

and the boundary conditions are given by

$$\nu_j \partial_s T^*_{-1/2,j} + \gamma T^*_{-1/2,j} = \left(K_{\parallel} \left(T^n_{-1/2,j} \right)^{5/2} - \nu_j \right) \partial_s T^n_{-1/2,j},$$

$$\nu_j \partial_s T^*_{N_s-1/2,j} - \gamma T^*_{N_s-1/2,j} = \left(K_{\parallel} \left(T^n_{N_s-1/2,j} \right)^{5/2} - \nu_j \right) \partial_s T^n_{N_s-1/2,j}.$$

This scheme is stable if the constants ν_j satisfy

$$K_{\parallel} \| T^n_{\cdot,j} \|_{\infty}^{5/2} \leq \nu_j, \; j = 0, \ldots, N_r - 1.$$

Numerical results I

In following simulations, we use the values of parameters as

$K_{\parallel}$	$K_{\perp}$	γ	$Q_{\perp}$	C_T	N_s	N_r
1	1e-3	2	4	5	100	100

The initial temperature is given by

$$T^0 = 0.5 Q_{\perp} r^2 - Q_{\perp} r + C_T.$$

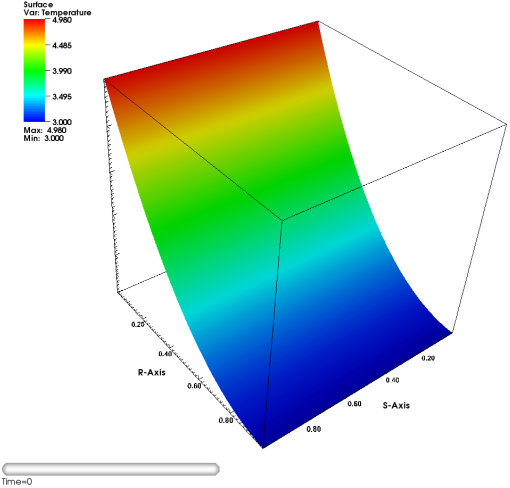
We use first the explicit scheme with time step

$$\Delta t^{n+1} \leq \min\left(\Delta t^n, \frac{\Delta r^2}{2K_{\perp}}, \frac{\Delta s^2}{2\max\left(K_{\parallel}\|T^n\|_{\infty}^{5/2}, \gamma\Delta s\right)}\right),$$

The total time is equal to 2, thus the computation time is 3305.52 s.

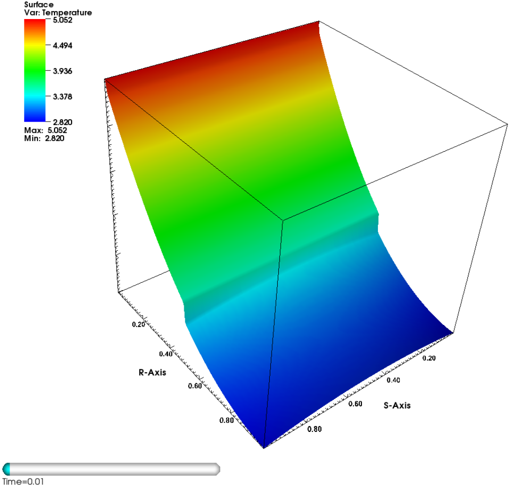
Numerical results II

2D temperature evolution: time=0



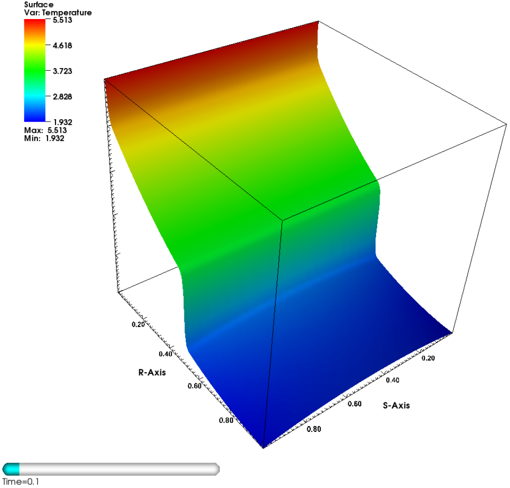
Numerical results III

2D temperature evolution: time=0.01



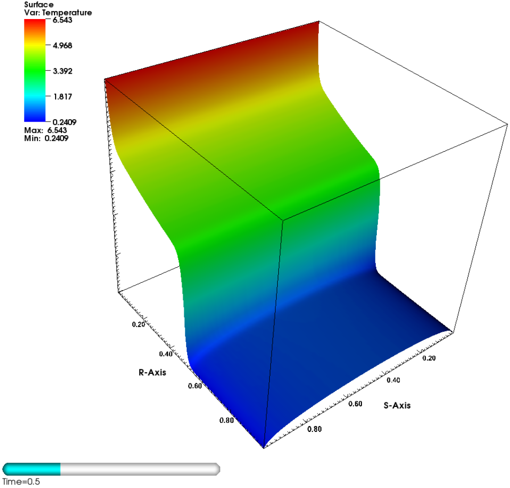
Numerical results IV

2D temperature evolution: time=0.1



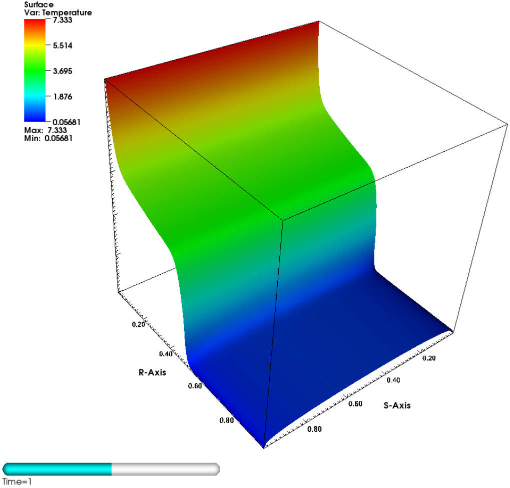
Numerical results V

2D temperature evolution: time=0.5



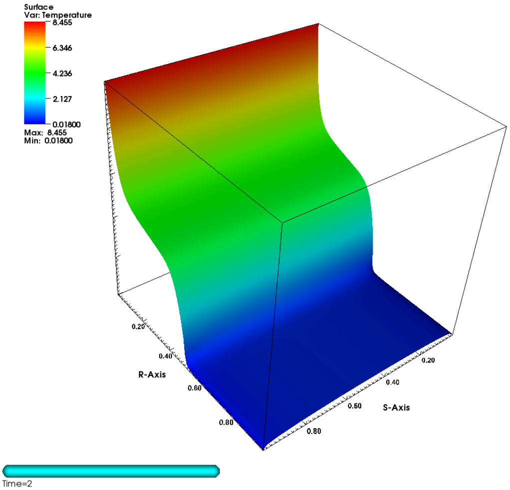
Numerical results VI

2D temperature evolution: time=1



Numerical results VII

2D temperature evolution: time=2



Numerical results VIII

The computation time of 2D problem:

Δt	0.1	0.01	0.001	0.0001
Implicit scheme	2.22	12.02	104.32	1327.50
Filbet-Jin scheme	1.04	5.02	37.07	403.63

Error estimation: $\|\cdot\| = \|T_{\text{explicit}} - T\|_2 / \|T_{\text{explicit}}\|_2$

Δt	0.1	0.01	0.001	0.0001
Implicit scheme	0.0048	4.4466e-4	4.3725e-5	4.3948e-6
Filbet-Jin scheme	0.0052	3.7325e-4	4.5247e-5	4.6455e-6

We aim to solve coupling problem $\mathcal{I}$ and $\mathcal{E}$

$$\mathcal{I} \quad \begin{cases} \partial_t T_i - \partial_s(K_{\|,i} T_i^{5/2} \partial_s T_i) - \partial_r(K_{\perp,i} \partial_r T_i) = \beta(T_i - T_e), \\ \text{B.C.} \end{cases}$$

$$\mathcal{E} \quad \begin{cases} \partial_t T_e - \partial_s(K_{\|,e} T_e^{5/2} \partial_s T_e) - \partial_r(K_{\perp,e} \partial_r T_e) = -\beta(T_i - T_e), \\ \text{B.C.} \end{cases}$$

where the source terms are equilibrium between ions and electrons. We use Lie splitting method in 3 steps, the first two steps are the same in 2D complete problem case, we have $\left(T_i^{**}, T_e^{**}\right)$. So it remains to solve

$$\begin{cases} \partial_t T_i = \beta(T_i - T_e), & \text{on } [0, \Delta t], \\ \partial_t T_e = -\beta(T_i - T_e), & \text{on } [0, \Delta t], \\ T_i(0) = T_i^{**}, \ T_e(0) = T_e^{**}. \end{cases}$$

We set finally

$$T_i^{n+1} := T_i(\Delta t),$$
$$T_e^{n+1} := T_e(\Delta t).$$

Conclusions and Perspectives

Conclusions

- The Filbet-Jin scheme is efficient for nonlinear problem
- The Filbet-Jin scheme combining with the splitting method is efficient for 2D problem

Work in progress

- Terminate the coupling problem

Thank you for your attention!