EMAFF: MAgnetic Equations with FreeFem++

The Grad-Shafranov equation
&
The Current Hole

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Outline

Outline

FreeFem++

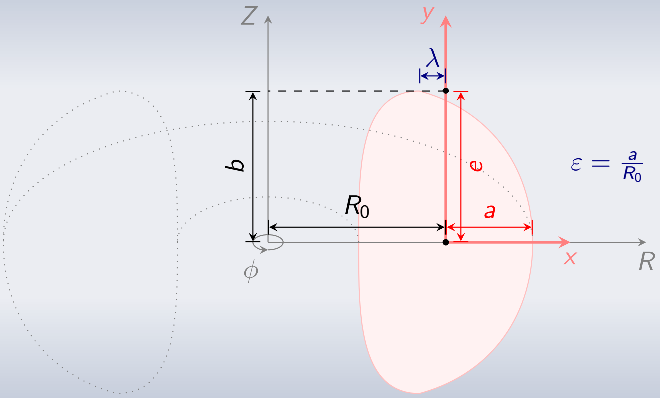
- FreeFem++[1] is a software to solve partial differential equations numerically, based on finite element methods.

- For the moment this platform is restricted to the numerical simulations of problems which admit a variational formulation.

- Our goal will be to evaluate the FreeFem++ tool on basic magnetic equations arising in Fusion Plasma.

[1] http://www.freefem.org/ff++/index.htm

Geometry configuration



$\varepsilon = \frac{a}{R_0}$

Reduced Resistive MHD

$$
\begin{cases}
\dfrac{\partial \psi}{\partial t} = (1+\varepsilon x)[\psi, \varphi] + \eta(J - J_c), \\
\dfrac{\partial \omega}{\partial t} = 2\varepsilon\omega \dfrac{\partial \varphi}{\partial y} + (1+\varepsilon x)[\omega, \varphi] + \dfrac{1}{1+\varepsilon x}[\psi, J] + \nu \Delta_\perp \omega, \\
J = \Delta^* \psi, \\
\omega = \Delta_\perp \varphi.
\end{cases}
$$

- ψ is the magnetic flux,
- φ is the velocity potential,
- J is the toroidal current density,
- ω is the vorticity,
- J_c is the non-ohmic driven current density that sets a constant profile likely to be perturbed with fluctuations,
- η stands for the resistivity,
- ν is the viscosity,
- $[\cdot, \cdot]$ is the Poisson brackets
- Δ^* is the Grad-Shafranov operator
- $\Delta_\perp$ is the laplacian restricted to the poloidal section.

EMAFF Project

- EMAFF-I: the Grad-Shafranov equation

$$-\Delta^* \psi = R^2 \frac{\mathrm{d}p}{\mathrm{d}\psi} + F \frac{\mathrm{d}F}{\mathrm{d}\psi};$$

- EMAFF-II: the current hole in cylindrical case (*i.e.* $\varepsilon = 0$)

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = [\psi, \varphi] + \eta (J - J_c), \\ \dfrac{\partial \omega}{\partial t} = [\omega, \varphi] + [\psi, J] + \nu \Delta_\perp \omega, \\ J = \Delta^* \psi, \\ \omega = \Delta_\perp \varphi. \end{cases}$$

Outline

The Grad-Shafranov Equation

In cylindrical coordinates (R, Z)

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}=-\left(R^2\frac{\mathrm{d}p}{\mathrm{d}\psi}+F\frac{\mathrm{d}F}{\mathrm{d}\psi}\right).$$

Soloviev equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}\psi}=\alpha=\mathrm{cst},\quad F\frac{\mathrm{d}F}{\mathrm{d}\psi}=\beta=\mathrm{cst}$$

In Cartesian coordinates (x, y)

$$\frac{1}{a^2}\left((1+\varepsilon x)\ \mathrm{div}\left(\frac{\nabla\psi}{1+\varepsilon x}\right)\right)=-\left(\alpha R_0^2(1+\varepsilon x)^2+\beta\right).$$

Example I - Soloviev equilibrium

Compute the magnetic flux ψ solution of the PDE

$$\frac{1}{R}\frac{\partial \psi}{\partial R} - \frac{\partial^2 \psi}{\partial R^2} - \frac{\partial^2 \psi}{\partial Z^2} = f_0(R^2 + R_0^2)$$

on the domain Ω defined by

$$\partial\Omega = \{(R, Z) \mid R = R_0\sqrt{1 + \tfrac{2a\cos\alpha}{R_0}},\ Z = aR_0 \sin\alpha,\ \alpha = 0 : 2\pi\}.$$

Example I - Soloviev equilibrium

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Analytical solution

$$\psi(R, Z) = \frac{f_0 R_0^2 a^2}{2}\left(1 - \left(\frac{Z}{a}\right)^2 - \left(\frac{R - R_0}{a} + \frac{(R - R_0)^2}{2aR_0}\right)^2\right).$$

Example I – Soloviev equilibrium

Computational parameters

- $a = 0.5$,
- $f_0 = 1$,
- $R_0 = 1$;

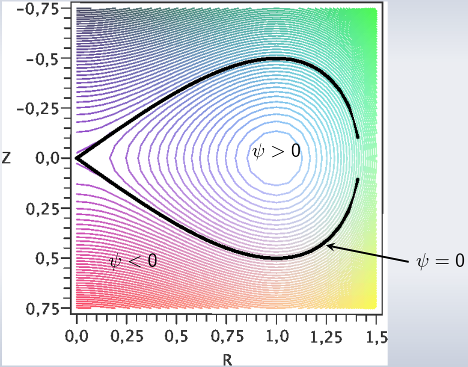
then

$$\psi(R, Z) = \frac{R^2}{4} - \frac{R^4}{8} - \frac{Z^2}{2}$$

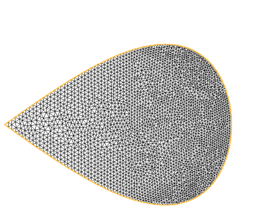
solution of the Grad-Shafranov equation

$$-\Delta_c^* \psi = R^2 + 1$$

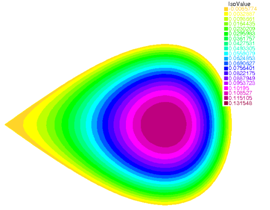
and $\psi|_{\partial\Omega} = 0$.



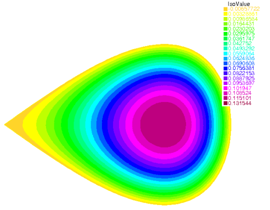
Example I – Soloviev equilibrium



Mesh with $N_{\partial\Omega} = 200$ elements on the border.



Magnetic flux ψ_{ex}.



Computed magnetic flux ψ with $\mathbb{P}^1$ finites elements type.

$$E(\psi) := \frac{\|\psi - \psi_{\text{ex}}\|_{L^2}}{\|\psi_{\text{ex}}\|_{L^2}} = 0.000145274$$

Example II - Soloviev equilibrium

Compute the magnetic flux ψ solution of the PDE

$$-\frac{\partial}{\partial x}\left(\frac{1}{1+\varepsilon x}\frac{\partial \psi}{\partial x}\right)+\frac{\partial^2\psi}{\partial y^2}=\left(\alpha R_0^2(1+\varepsilon x)^2+\beta\right)\frac{a^2}{(1+\varepsilon x)}$$

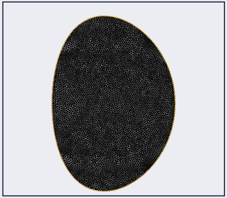
on the domain Ω defined by $\partial\Omega=\{(x,y)\in\mathbb{R}^2|\psi(x,y)=0\}$.

Analytical solution

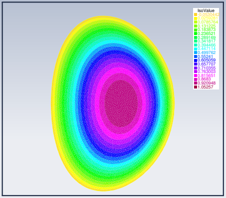
$$\begin{aligned}\psi(x,y)=1-&\left(x-\frac{\varepsilon}{2}(1-x^2)\right)^2\\&-\left(\left(1-\frac{\varepsilon^2}{4}\right)(1+\varepsilon x)^2+\lambda x\left(1+\frac{\varepsilon}{2}x\right)\right)\left(\frac{a}{b}y\right)^2\end{aligned}$$

$$\alpha=\frac{4(a^2+b^2)\varepsilon+a^2(2\lambda-\varepsilon^3)}{2R_0^2\varepsilon a^2 b^2},\qquad \beta=-\frac{\lambda}{b^2\varepsilon}.$$

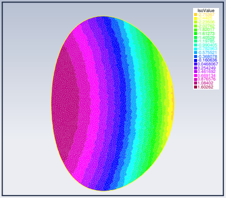
Example II – Soloviev equilibrium



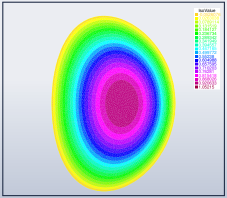
Mesh for $N_{\partial\Omega} = 128$.



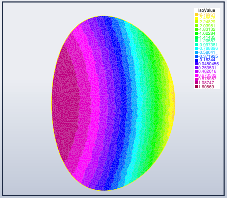
ψ_{ex}



$\frac{\partial \psi_{\text{ex}}}{\partial x}$



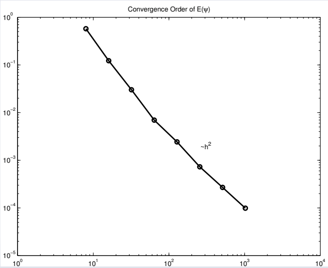
ψ

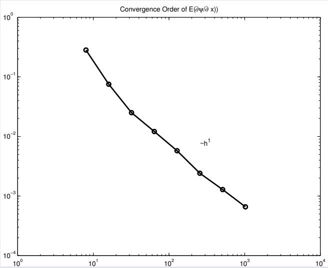


$\frac{\partial \psi}{\partial x}$

Example II - Soloviev equilibrium

$$E(\xi) := \frac{\|\xi - \xi_{ex}\|_{L^2}}{\|\xi_{ex}\|_{L^2}} = \sqrt{\frac{\int_\Omega (\xi - \xi_{ex})^2 \, d\Omega}{\int_\Omega (\xi_{ex})^2 \, d\Omega}}$$





$$N_{\partial\Omega} \mapsto E(\psi)$$

$$N_{\partial\Omega} \mapsto E\left(\frac{\partial}{\partial x}\psi\right)$$

$\mathbb{P}^1$ elements provide 2 convergence rate for the magnetic flux while gradients are approximated by one order of magnitude less.

Example III - polynomial (nonlinear) RHS

- The equation

$$-\Delta_c^* \psi(R,Z) = G(R,Z;\psi)$$

with RHS $G(R,Z;\psi)$ polynomial in ψ (nonlinear).

- Iterative method idea: numerical linearization of the RHS.

- The general scheme

$$\begin{cases} \psi_0 \text{ given} \\ -\Delta_c^* \psi_{k+1} = \psi_{k+1} G_1(R,Z;\psi_k) + G_2(R,Z;\psi_k) \end{cases}$$

Example III – polynomial (nonlinear) RHS

The test case

$$\psi_{ex} = \frac{6}{9aR^2 + k_1(z + c_1)^2}$$

solution of

$$-\Delta_c^* \psi = \psi^2 \left(k_1 + 2a(k_1 - 9a)R^2 \psi \right)$$

Schemes

- the start point

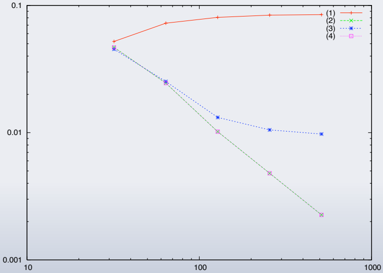
$$\psi_0 = \psi_{ex} + \frac{1}{2} cos(R),$$

- the discretization of the RHS

$$\text{RHS} = k_1 \psi_{k+1}^{\beta} {\psi_k}^{2-\beta} + 2a(k_1 - 9a)R^2 \psi_{k+1}^{\delta} {\psi_k}^{3-\delta}.$$

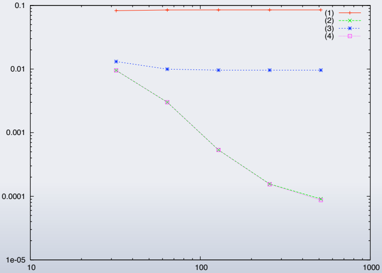
Example III – polynomial (nonlinear) RHS

	(1)	(2)	(3)	(4)
β	0	0	1	1
δ	0	1	0	1



$$N_{\partial\Omega} \mapsto \frac{\|\psi - \psi_{\text{ex}}\|_{H^1(\Omega)}}{\|\psi_{\text{ex}}\|_{H^1(\Omega)}}$$

with elements $\mathbb{P}^1$



$$N_{\partial\Omega} \mapsto \frac{\|\psi - \psi_{\text{ex}}\|_{H^1(\Omega)}}{\|\psi_{\text{ex}}\|_{H^1(\Omega)}}$$

with elements $\mathbb{P}^2$

Outline

The equations

Reduced model with simplified geometry ($\varepsilon = 0$, $x \in \Omega$, $t \in [0, T]$)

$$\begin{cases} \partial_t \psi = [\psi, \varphi] + \eta(J - J_c), \\ \partial_t \omega = [\omega, \varphi] + [\psi, J] + \nu \Delta \omega, \\ J = \Delta \psi, \\ \omega = \Delta \varphi \end{cases}$$

with

- Poisson brackets

$$[a, b] := \frac{\partial a}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a}{\partial x_2} \frac{\partial b}{\partial x_1}$$

- initial conditions

$$\begin{cases} \varphi(0, x) = \omega(0, x) = 0, & x \in \Omega, \\ J(0, x) = J_c, & x \in \Omega, \\ \psi(0, x) = \Delta^{-1} J_c, & x \in \Omega; \end{cases}$$

- boundary conditions

$$\varphi(t, x) = \omega(t, x) = \psi(t, x) = J(t, x) = 0, \qquad x \in \partial\Omega, \ t \in [0, T].$$

The FreeFem++ space discretization

- Finite elements with $\mathbb{P}^1$ (order 2) or $\mathbb{P}^2$ (order 3).
- Variational formulation:

$$\mathbb{A}\mathbf{u} = \mathbf{f}$$

solved by $\mathbf{u} \in V_h$ ($\mathbb{P}^1$ or $\mathbb{P}^2$) and

$$\langle \mathbf{v}, \mathbb{A}\mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{f} \rangle, \qquad \forall \mathbf{v} \in_h$$

Discretization in time

- Finite elements favor implicit time discretization.
- Crank-Nicholson (order 2) with linearization

$$\partial_t u = F(u) \quad \longrightarrow \quad \frac{u_{n+1} - u_n}{\delta t} = \frac{1}{2}\left(F(u_n) + F(u_{n+1})\right)$$

with $u_{n+1} = u_n + \delta u$, then we have to solve

$$\underbrace{\left(Id - \frac{\delta t}{2} \nabla F(u_n) \right)}_{\mathbb{M}} \delta u = \delta t \; F(u_n).$$

Matricial Formulation - I

First choice for our problem:

$$\begin{pmatrix}\psi\\ \varphi\end{pmatrix}_{n+1} = \begin{pmatrix}\psi\\ \varphi\end{pmatrix}_{n} + \begin{pmatrix}\delta\psi\\ \delta\varphi\end{pmatrix}$$

which implies to solve

$$\mathbb{M}\begin{pmatrix}\delta\psi\\ \delta\varphi\end{pmatrix} = \delta t\begin{pmatrix}[\psi_n,\varphi_n]+\eta\Delta\psi_n-\eta J_c\\ [\Delta\varphi_n,\varphi_n]+[\psi_n,\Delta\psi_n]+\nu\Delta^2\varphi_n\end{pmatrix}$$

with

$$\mathbb{M} := \begin{pmatrix} Id+\frac{\delta t}{2}[\varphi_n,\cdot]-\frac{\delta t\,\eta}{2}\Delta & -\frac{\delta t}{2}[\psi_n,\cdot]\\ -\frac{\delta t}{2}[\psi_n,\Delta\cdot]+\frac{\delta t}{2}[\Delta\psi_n,\cdot] & \Delta-\frac{\delta t}{2}[\Delta\varphi_n,\cdot]+\frac{\delta t}{2}[\Delta\varphi_n,\cdot]-\frac{\delta t\,\nu}{2}\Delta^2\end{pmatrix}$$

$\Delta^2 \implies$ we can not use $\mathbb{P}^1$ elements

$\downarrow$

we introduce $J=\Delta\psi$ and $\omega=\Delta\varphi$ to obtain a new Matricial Formulation.

Matricial Formulation - II

$$\mathbb{M}\begin{pmatrix}\delta\psi\\ \delta\varphi\\ \delta J\\ \delta\omega\end{pmatrix} = \delta t\begin{pmatrix}[\psi_n,\varphi_n] + \eta(J_n - J_c)\\ [\omega_n,\varphi_n] + [\psi_n, J_n] + \nu\Delta\omega_n\\ 0\\ 0\end{pmatrix}$$

with

$$\mathbb{M} :\ = \begin{pmatrix} Id + \frac{\delta t}{2}[\varphi_n,\cdot] & -\frac{\delta t}{2}[\psi_n,\cdot] & -\frac{\delta t\,\eta}{2} Id & 0\\ \frac{\delta t}{2}[J_n,\cdot] & -\frac{\delta t}{2}[\omega_n,\cdot] & -\frac{\delta t}{2}[\psi_n,\cdot] & Id + \frac{\delta t}{2}[\varphi_n,\cdot] - \frac{\delta t\,\nu}{2}\Delta\\ 0 & -\Delta & 0 & Id\\ -\Delta & 0 & Id & 0\end{pmatrix}$$

Computational Tests

- Tests I:
  - $\mathbb{P}^1$: see the video
  - $\mathbb{P}^2$: still in progress
- Tests II:
  - $\mathbb{P}^1$ (G. Huysmans paper): see the video
  - $\mathbb{P}^2$: see the video
- Tests III:
  - $\mathbb{P}^1$ (J_c initial Laplace): see the video

Outline

Conclusion – Perspectives

GS-equation

- ✓ FreeFem++ is very easy to handle and adapted to solve this kind of problems in general geometry,
- ✓ good agreement with the (limited!) literature,
- ✗ Work in progress: polynomial nonlinear cases
  - several solutions: how to select the physical one?
  - initial condition and FE type: how to choose them?

The current-Hole

- ✓ effective simulation,
- ✓ no need of any initial perturbation (included in space approximation),
- ✓ different geometries,
- ✓ many test cases with FreeFem++.