Quest for Osher-Type Riemann Solver for Ideal MHD Equations

Project Proposal CEMRACS 2010

Hervé Guillard, Barry Koren, Boniface Nkonga

CIRM, July 19 – August 27, 2010

Wine guide to Riemann solvers

Godunov: Produced from vines of great antiquity. Full-bodied vintage to satisfy the most demanding palate. Modern viniculture is able to deliver most of its features at substantially lower cost though.

Wine guide to Riemann solvers

Godunov: Produced from vines of great antiquity. Full-bodied vintage to satisfy the most demanding palate. Modern viniculture is able to deliver most of its features at substantially lower cost though.

Roe: Moderately robust and finely flavored. Its variable density makes it unique. Be warned, though, that its method of manufacture may cause very local headache in some states.

Wine guide to Riemann solvers

Godunov: Produced from vines of great antiquity. Full-bodied vintage to satisfy the most demanding palate. Modern viniculture is able to deliver most of its features at substantially lower cost though.

Roe: Moderately robust and finely flavored. Its variable density makes it unique. Be warned, though, that its method of manufacture may cause very local headache in some states.

Harten – Lax – van Leer: Robust wine that may be served on most occasions, but the omission of certain traditional ingredients leads to cloudiness and a sticky aftertaste.

Wine guide to Riemann solvers

Godunov: Produced from vines of great antiquity. Full-bodied vintage to satisfy the most demanding palate. Modern viniculture is able to deliver most of its features at substantially lower cost though.

Roe: Moderately robust and finely flavored. Its variable density makes it unique. Be warned, though, that its method of manufacture may cause very local headache in some states.

Harten – Lax – van Leer: Robust wine that may be served on most occasions, but the omission of certain traditional ingredients leads to cloudiness and a sticky aftertaste.

Osher: Robust, smooth top-quality wine with a high proof content. From the best Californian winery. Not a cheap wine to speed the gaiety. No artificial additives have been used. In my opinion the best buy!

Properties Osher scheme

- Strengths
  - Closest similarity to Godunov
  - Consistent boundary-condition treatment
  - Consistent source-term treatment
  - Continuous differentiability

Properties Osher scheme

- Strengths
  - Closest similarity to Godunov
  - Consistent boundary-condition treatment
  - Consistent source-term treatment
  - Continuous differentiability
- Challenges
  - Construction
  - Computational intensity

Osher scheme in a nutshell

Consider $\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$ and suppose $f^{+}(q)$ and $f^{-}(q)$ exist such that:

$$f(q) = f^{+}(q) + f^{-}(q).$$

Then, a natural approximate Riemann solver is:

$$F(q^{l}, q^{r}) = f^{+}(q^{l}) + f^{-}(q^{r}),$$

which can also be written as:

$$F(q^{l}, q^{r}) = f(q^{l}) - f^{-}(q^{l}) + f^{-}(q^{r}) = f(q^{l}) + \int_{q^{l}}^{q^{r}} \frac{\mathrm{d}f^{-}}{\mathrm{d}q} \mathrm{d}q,$$

$$\text{or}: \quad F(q^{l}, q^{r}) = f(q^{r}) - f^{+}(q^{r}) + f^{+}(q^{l}) = f(q^{r}) - \int_{q^{l}}^{q^{r}} \frac{\mathrm{d}f^{+}}{\mathrm{d}q} \mathrm{d}q.$$

Osher scheme in a nutshell

Consider $\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$ and suppose $f^+(q)$ and $f^-(q)$ exist such that:

$$f(q) = f^+(q) + f^-(q).$$

Then, a natural approximate Riemann solver is:

$$F(q^l, q^r) = f^+(q^l) + f^-(q^r),$$

which can also be written as:

$$F(q^l, q^r) = f(q^l) - f^-(q^l) + f^-(q^r) = f(q^l) + \int_{q^l}^{q^r} \frac{\mathrm{d}f^-}{\mathrm{d}q}\mathrm{d}q,$$

$$\text{or}: \quad F(q^l, q^r) = f(q^r) - f^+(q^r) + f^+(q^l) = f(q^r) - \int_{q^l}^{q^r} \frac{\mathrm{d}f^+}{\mathrm{d}q}\mathrm{d}q.$$

Osher's elegant choice: integration along eigenvectors.

Task 1

Construct Osher scheme for 1D ideal MHD equations,
$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = s(q)$, with Ken Powell's source terms:

$$q = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ B_1 \\ B_2 \\ B_3 \\ e \end{pmatrix}, \; f(q) = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} - B_1^2 \\ \rho u_1 u_2 - B_1 B_2 \\ \rho u_1 u_3 - B_1 B_3 \\ 0 \\ B_2 u_1 - B_1 u_2 \\ B_3 u_1 - B_1 u_3 \\ u_1 \left(e + p + \frac{1}{2}\mathbf{B}\cdot\mathbf{B}\right) - B_1 \mathbf{u}\cdot\mathbf{B} \end{pmatrix}, \; s(q) = \mathrm{div}\mathbf{B} \begin{pmatrix} 0 \\ -B_1 \\ -B_2 \\ -B_3 \\ u_1 \\ u_2 \\ u_3 \\ \mathbf{u}\cdot\mathbf{B} \end{pmatrix}.$$

K.G. POWELL, An approximate Riemann solver for magnetohydrodynamics (that works in more than one dimension), *CWI Report NM-R9407*, Amsterdam (1994).

Task 2

Construct Osher scheme for 1D ideal MHD equations,
$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = s(q)$, with 'divergence cleaning' terms:

$$q = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ B_1 \\ B_2 \\ B_3 \\ e \\ \psi \end{pmatrix}, \quad f(q) = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} - B_1^2 \\ \rho u_1 u_2 - B_1 B_2 \\ \rho u_1 u_3 - B_1 B_3 \\ \psi \\ B_2 u_1 - B_1 u_2 \\ B_3 u_1 - B_1 u_3 \\ u_1\left(e + p + \frac{1}{2}\mathbf{B}\cdot\mathbf{B}\right) - B_1 \mathbf{u}\cdot\mathbf{B} \\ c_h^2 B_1 \end{pmatrix}, \quad s(q) = \begin{pmatrix} 0 \\ -\mathrm{div}\mathbf{B} B_1 \\ -\mathrm{div}\mathbf{B} B_2 \\ -\mathrm{div}\mathbf{B} B_3 \\ 0 \\ 0 \\ 0 \\ -B_1 \frac{\partial \psi}{\partial x} \\ -\frac{c_h^2}{c_p^2}\psi \end{pmatrix}.$$

A. DEDNER ET AL., Hyperbolic divergence cleaning for the MHD equations, *J. Comput. Phys.*, **175**, 645–673 (2002).

Success !