Interfaces, level set methods and applications

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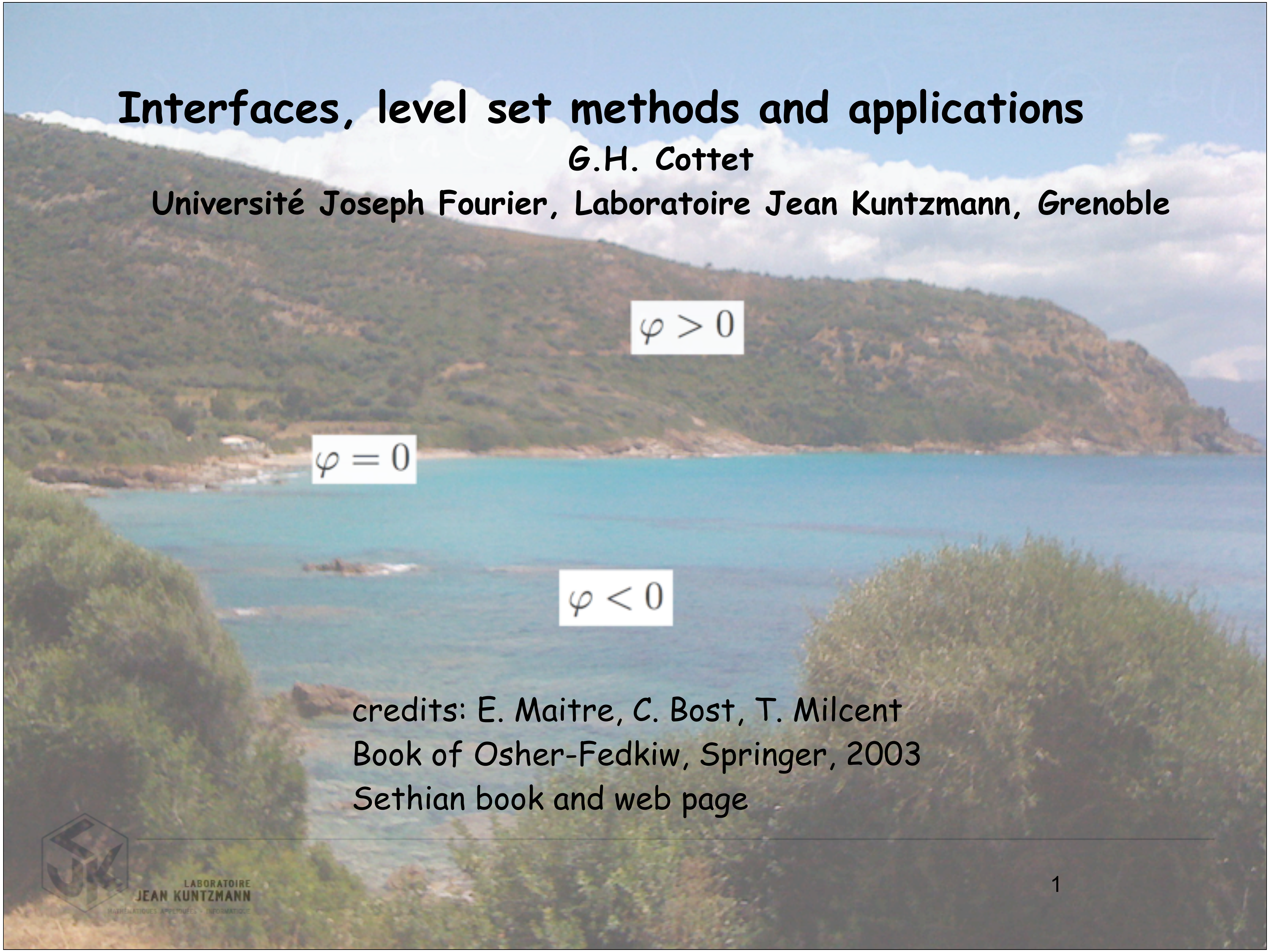

credits: E. Maitre, C. Bost, T. Milcent

Book of Osher-Fedkiw, Springer, 2003

Sethian book and web page

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A few references (in addition to the books of Osher-Fedkiw and Sethian and the references therein) more or less by chronological order.
Pdf files for publications with my names can be found in
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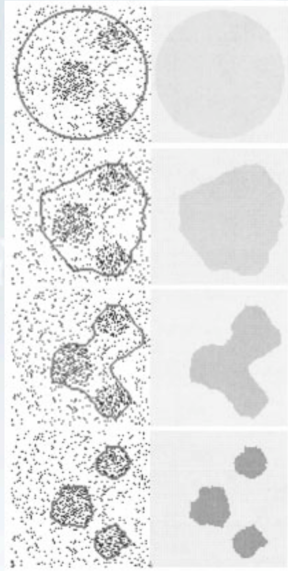
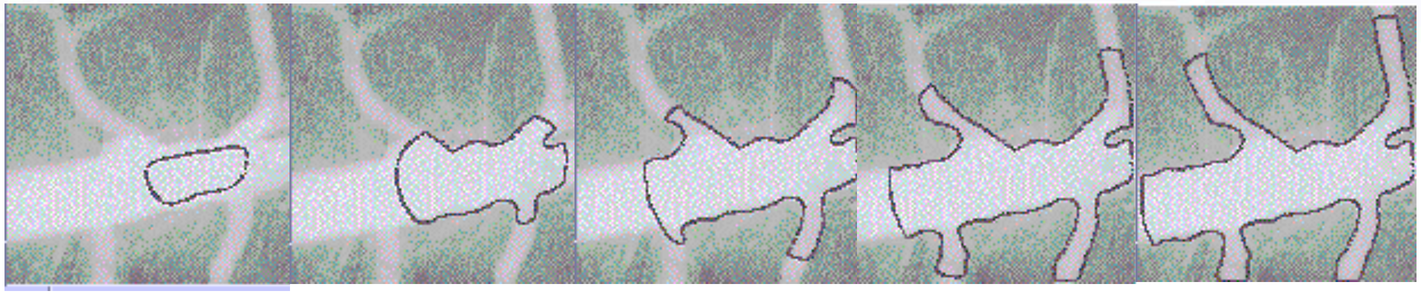
Goal:
present level set methodology, and applications to image processing (classical stuff) and fluid-structure (more recent stuff)
try to emphasize applications to biology

Interfaces appear from different view-points

numerical tools to represent, identify structures
or
physical, material “active” elements

First case: “virtual” interfaces





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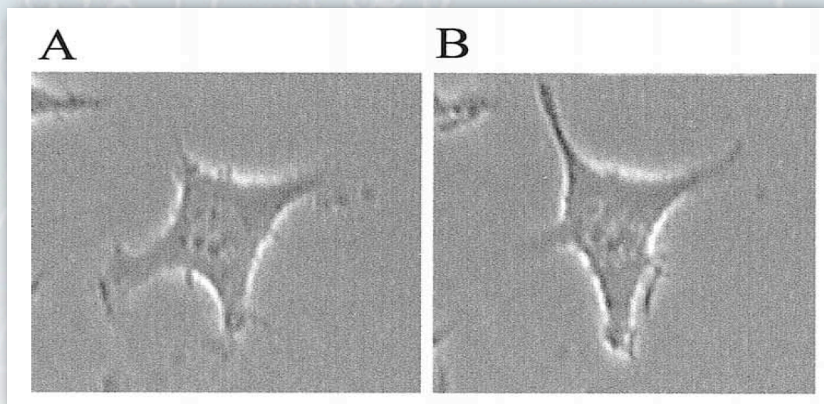
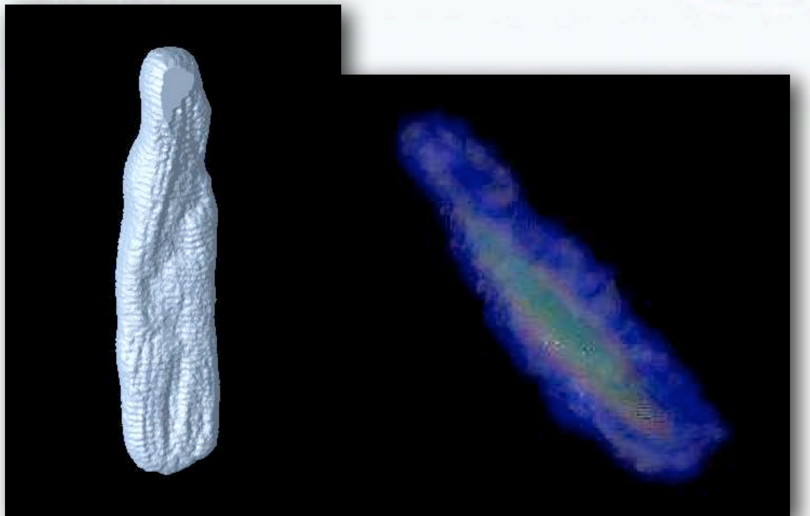
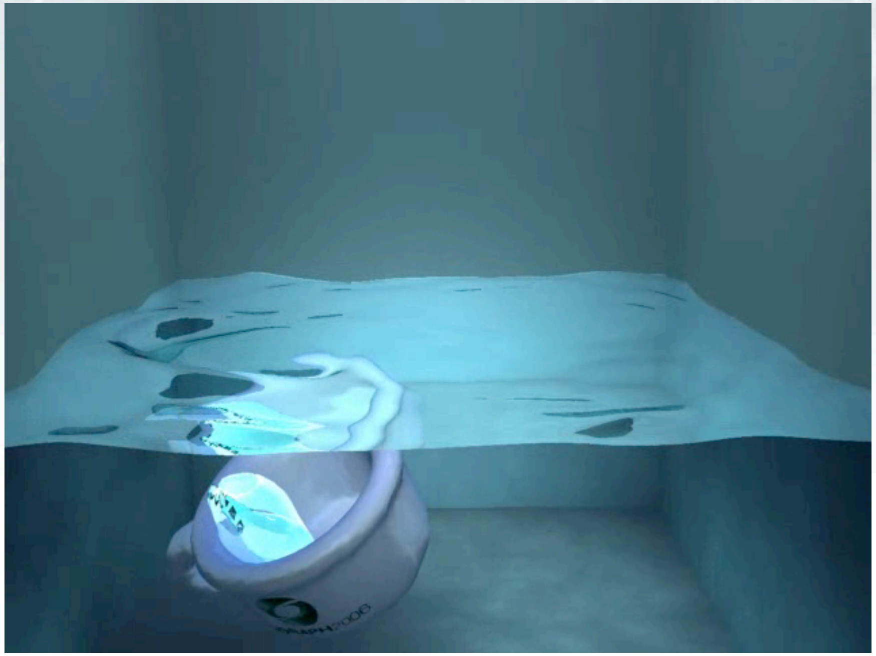
Second case: material interfaces





Mixing of virtual and physical interfaces geometrical and physical modeling for image synthesis



Outline :

lecture 1 : general definitions, geometrical properties, surface integrals

lecture 2 : image processing 1: diffusion models and curvature motion

lecture 3 : image processing 2 : contours and snakes

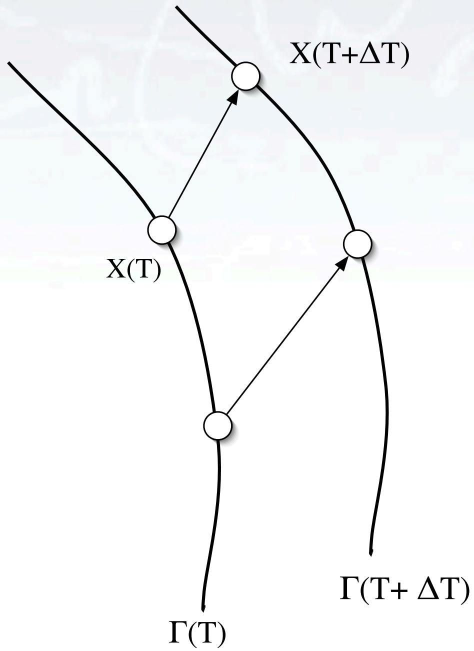
lecture 4 : fluid-structure interaction: membranes and biological vesicles

lecture 5 : fluid-structure interaction: general case and elastic cells

lecture 6 : dealing with stability and collisions

Interface tracking vs interface capturing

Interested in *material* - or *lagrangian* - interfaces moving with a given velocity **u**



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Two kinds of interface representation:

•explicit parametrization coupled with motion of individual markers on surface : interface tracking

•implicit representation and solution of a PDE “everywhere”: interface capturing

interface tracking requires interpolation, “surgery” and topology decisions
level set models avoid these problems (to some extent) but are less intuitive.

Level set method (implicit surface method):

Theorem:
if $\Gamma(t)$ is the level set of a smooth scalar function Φ ($\Gamma(t)= \{ \Phi(\bullet,t) = 0\}$) and if Φ satisfies

$$\Phi_t + (\mathbf{u}\bullet\nabla)\Phi = 0,$$

then $\Gamma(t+\Delta t)= \{ \Phi(\bullet,t+\Delta t) = 0\}$

Proof:
if $d\mathbf{X}/dt=\mathbf{u}(\mathbf{X},t)$, $d/dt(\Phi(\mathbf{X}(t),t)=0$ and $\Phi(\mathbf{X}(t),t) = 0 \Rightarrow \Phi(X(t+\Delta t),t+\Delta t) = 0$

Consequence: material interface can be captured as time evolution of a given level set for the solution of an advection PDE.

Question: what kind of geometrical information can be retrieved from Φ (if possible in a simpler way than a parametrization)

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First information: normal to the interface given by the gradient of the level set function

unit normal vector : $\mathbf{n}=\nabla\Phi/|\nabla\Phi|$

Second information: curvature given by the divergence of the normal:

κ= div **n**

$$\kappa(\phi) = -\frac{\phi_y^2\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy}}{(\phi_x^2 + \phi_y^2)^{3/2}}$$

Important remark: interface motion defined up to any tangential velocity:

Equivalent equation for Φ:

$\Phi_t + \mathbf{u.n}(\mathbf{n}\bullet\nabla)\Phi = 0$

or

$\Phi_t + v\,|\nabla\Phi| = 0$: Hamilton Jacobi equation

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links with conservation laws:

consider 1-D case: $\Phi_t + H(\Phi_x)=0$

set $u=\Phi_x$

and differentiate HJ equation:

$u_t + H(u)_x = 0.$

consequence: singularity in Φ_x (kinks) can arise.

can expect strong variations in level set slopes

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many questions involve computing surface integrals on the interface

-> how to do that with a level set representation (without using a surface explicit parametrization) ?

simplest answer : regularize interface and rely on volume integrals

Theorem:

Let ζ a 1-D, even, mollifying function such that

$$\int_{-\infty}^{+\infty} \zeta(x)\, dx = 1 \qquad \int_{-\infty}^{+\infty} x^2\, \zeta(x)\, dx < +\infty$$

then: $$\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla\phi| \rightharpoonup \delta_{\{\phi=0\}} \quad dans\ \mathcal{M}(\mathbb{R}^d)$$

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Proof:

need to check that $$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla\phi|f(x)dx=\int_{\{\phi=0\}}f(x)d\sigma$$

by definition of ζ, ro any smooth function g : $$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{1}{\varepsilon}\zeta\left(\frac{r}{\varepsilon}\right)g(r)dr=g(0)$$

set : $$g(r)=\int_{\{\phi=r\}}fd\sigma$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{1}{\varepsilon}\zeta\left(\frac{r}{\varepsilon}\right)\int_{\{\phi=r\}}fd\sigma dr=\int_{\{\phi=0\}}fd\sigma$$

Assume for simplicity that ζ has compact support in [-1,+1]. Then

$$\int_{\mathbb{R}}\frac{1}{\varepsilon}\zeta\left(\frac{r}{\varepsilon}\right)\int_{\{\phi=r\}}fd\sigma dr=\int_{-\varepsilon}^{\varepsilon}\int_{\{\phi=r\}}\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)fd\sigma dr$$

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Lemma : $$\int_{|\phi(x)|<\eta} g(x)dx = \int_{-\eta}^{\eta} \int_{\phi(x)=\nu} g(x)|\nabla\phi|^{-1} d\sigma d\nu$$

$g = \zeta_\epsilon f \, |\nabla\phi|$ gives the answer.

Proof of lemma:

measure theory gives

$$\frac{d}{ds}\left(\int_{\phi>s} g(x)dx\right) = -\int_{\phi=s} g|\nabla\phi|^{-1} d\sigma \qquad \text{p.p. } s$$

set $s=\eta$ then use for ϕ and $-\phi$, integrate and sum.

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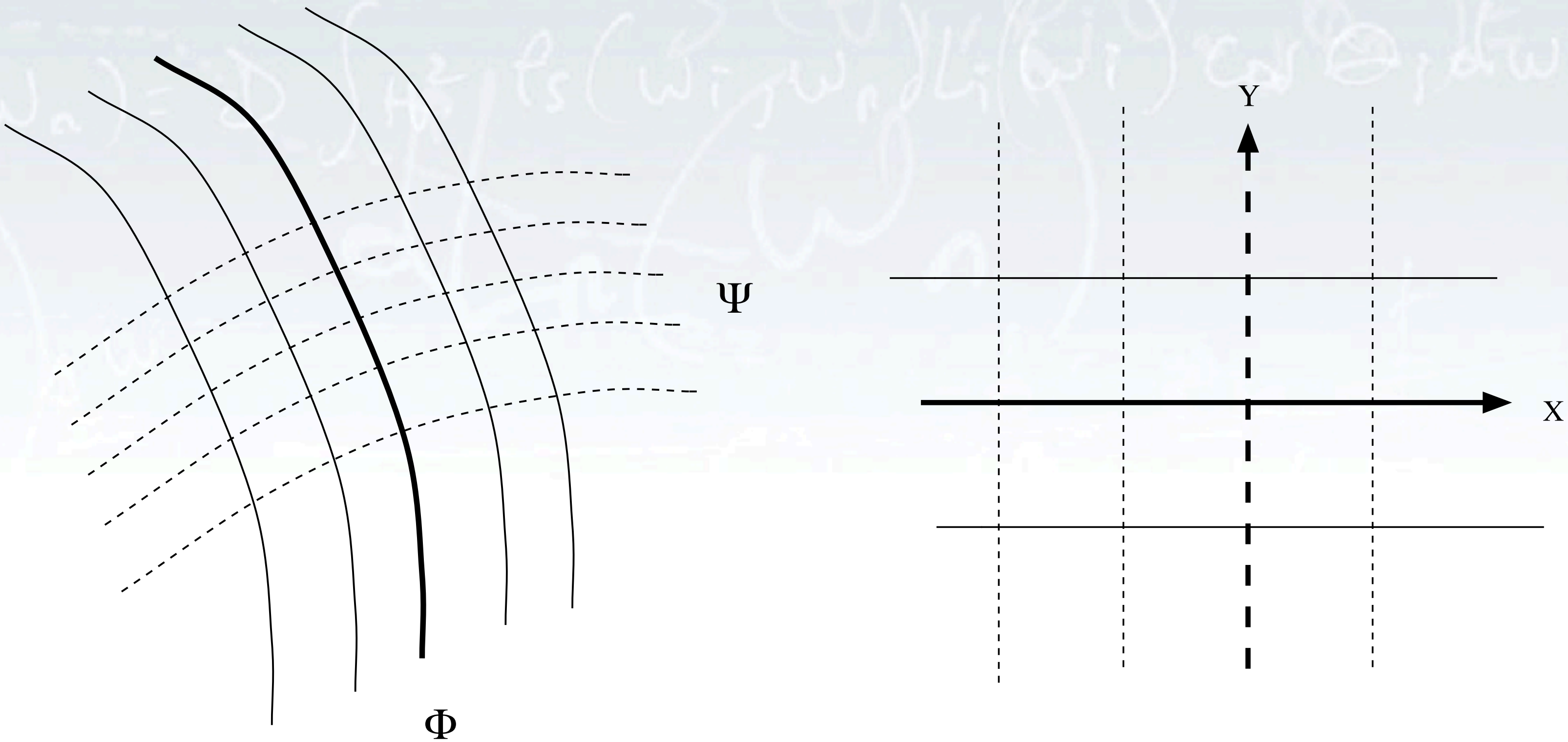
Other proof by change of variables using a transverse function ψ



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To be effective, regularization requires the control of the slope $|\nabla\phi|$:

- if $|\nabla\phi|$ is too small, too much smearing
- if $|\nabla\phi|$ is too large, needs many grid quadrature points to capture interface

$$\frac{\partial \phi}{\partial \tau} + \operatorname{sgn}(\phi_0)(|\nabla \phi| - 1) = 0$$

$$\frac{\partial \phi}{\partial \tau} + \operatorname{sgn}(\phi_0)\frac{\nabla \phi}{|\nabla \phi|} \cdot \nabla \phi = -\operatorname{sgn}(\phi_0)$$

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Other method:
solve $|\nabla\Phi|=1$
with upwind differencing
and propagate solution
from the lowest values

(fast marching method)

$$|\nabla u(x,y,z)| = F(x,y,z)$$

upwind differencing

$$\begin{bmatrix} \max(D^{-x}_{ijk}u, -D^{+x}_{ijk}u, 0)^2 + \\ \max(D^{-y}_{ijk}u, -D^{+y}_{ijk}u, 0)^2 + \\ \max(D^{-z}_{ijk}u, -D^{+z}_{ijk}u, 0)^2 \end{bmatrix}^{1/2} = F_{ijk}$$

allows to propagate u form low values (0 at the interface), to larger values (outward).

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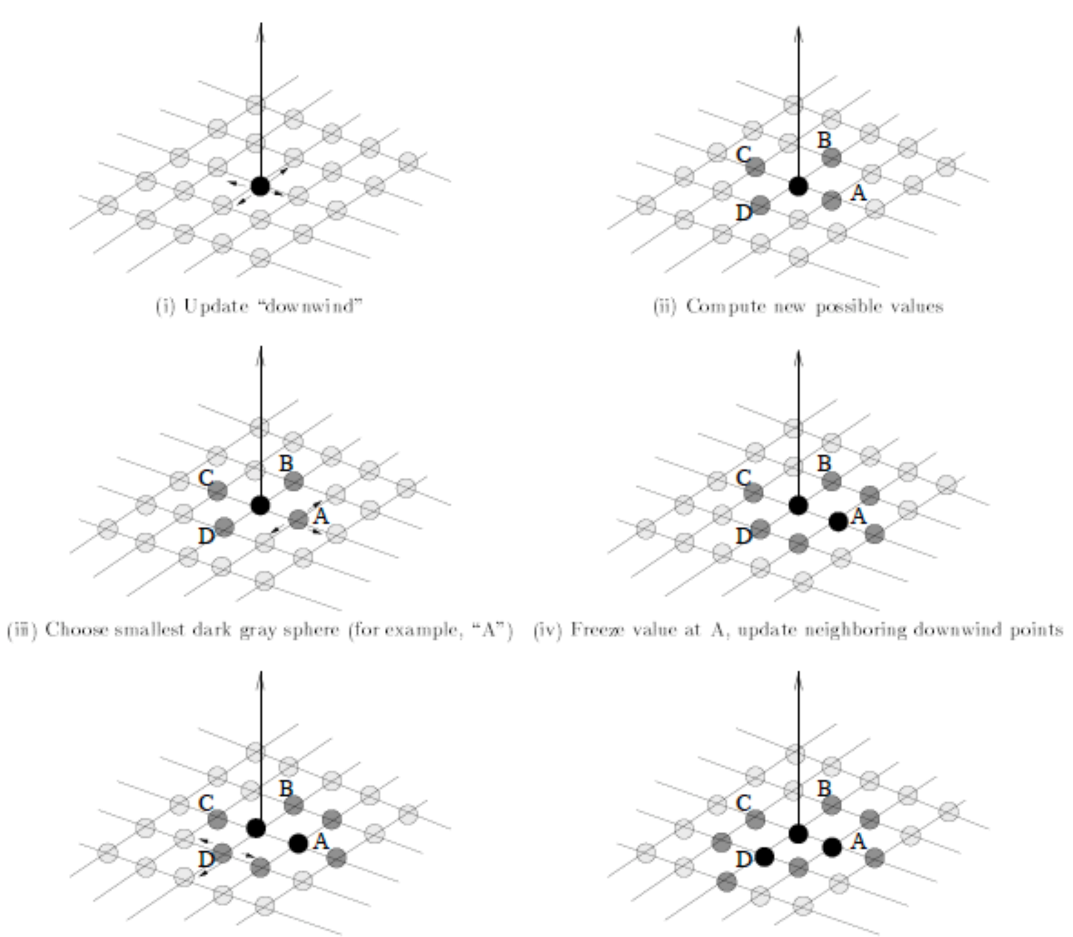

(i) Update "downwind"

(ii) Compute new possible values

(iii) Choose smallest dark gray sphere (for example, "A")

(iv) Freeze value at A, update neighboring downwind points

(v) Choose smallest dark gray sphere (for example, "D")

(vi) Freeze value at D, update neighboring downwind points

sketch of fast marching algorithm

Advantages:
distance to the interface is a nice information
makes the level set function better behaved for numerical approximations (*e.g.* to compute curvature)

Drawback :
moves the level sets (volume loss)

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Other approach: observe that Φ/|∇Φ| approximates the distance function, close to the interface

$$\phi(x - d(x)\nabla d) = 0 \quad \Rightarrow \quad \phi(x) - d(x)\nabla\phi \cdot \nabla d(x) + o(d(x)) = 0$$

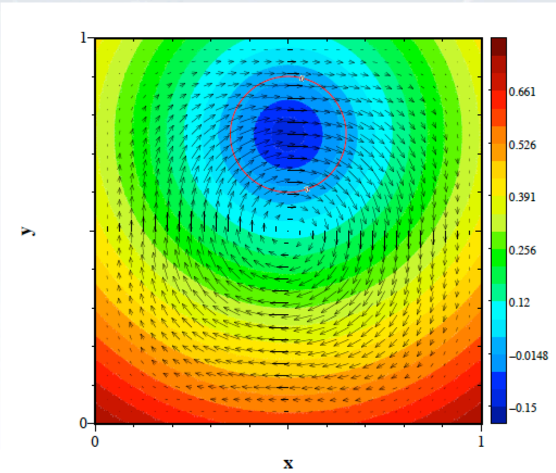
but $\nabla d(x) \simeq \frac{\nabla\phi}{|\nabla\phi|}(x)$ because d(x-nd(x)) = 0

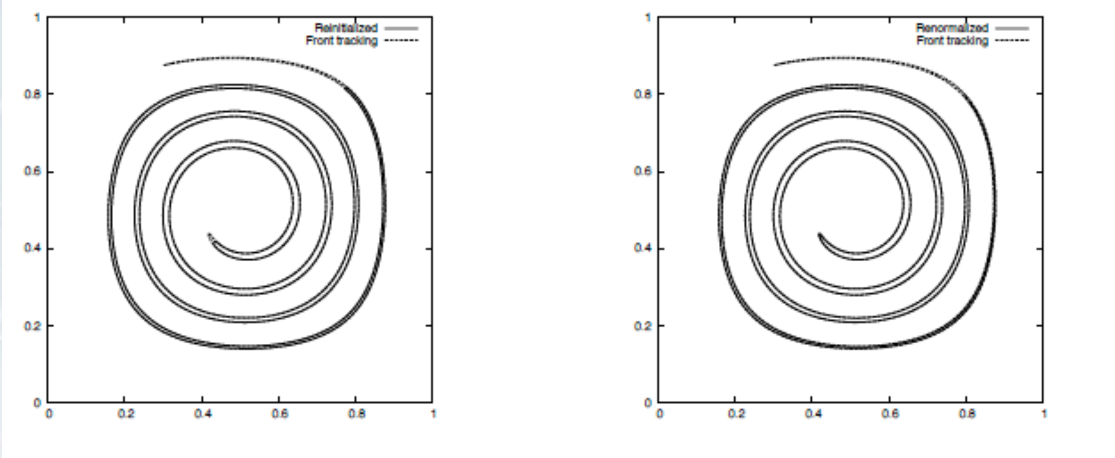
therefore

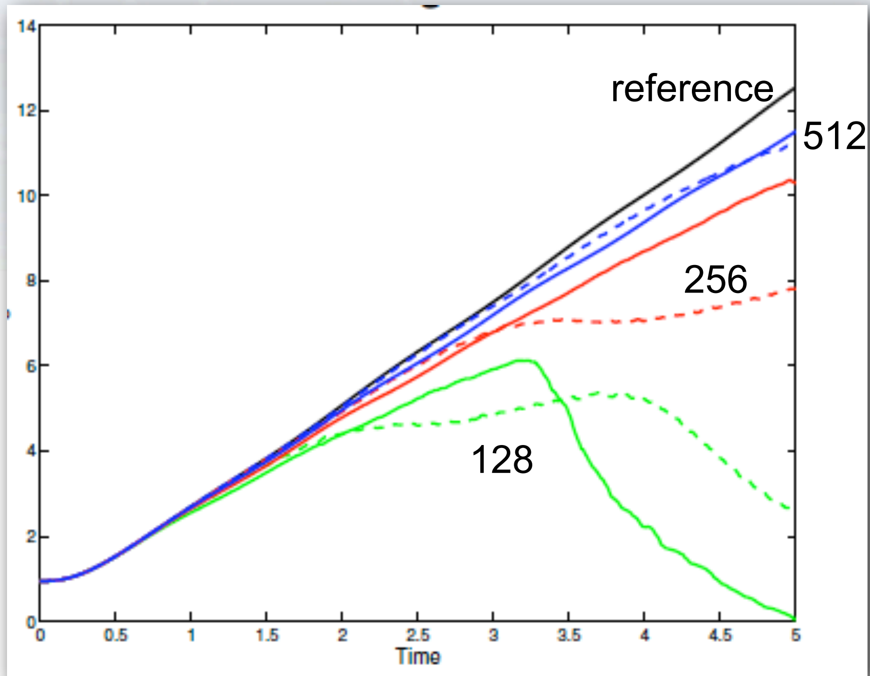
$$\phi(x) - d(x)|\nabla\phi|(x) + O(d(x)) = 0$$ and $$d(x) \approx \frac{\phi}{|\nabla\phi|}(x)$$

Consequence : can replace $\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla\phi|$ by $\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon|\nabla\phi|}\right)$

Illustration:
computation of the length of the interface in a spiraling circle







solid: with reinitialization
dotted: with renormalization

(with Weno scheme)

$$\varphi(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \qquad L = \int_\Omega \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)|\nabla\varphi| dxdy$$

Mesh Size	Smereka 1		Renormalization		Smereka 2	
0.2	9.38×10^{-3}		1.5×10^{-1}		2.68×10^{-3}	
0.1	2.23×10^{-3}	2.07	5×10^{-3}		5.49×10^{-4}	2.29
0.05	8.12×10^{-4}	1.46	1.3×10^{-3}	1.9	1.32×10^{-4}	2.05
0.025	2.71×10^{-4}	1.58	3×10^{-4}	2.1	2.90×10^{-5}	2.18
0.0125	7.58×10^{-5}	1.83	8×10^{-5}	1.9	7.79×10^{-6}	1.90
0.00625	3.04×10^{-5}	1.32	2×10^{-5}	2	1.84×10^{-6}	2.08

Lecture 2

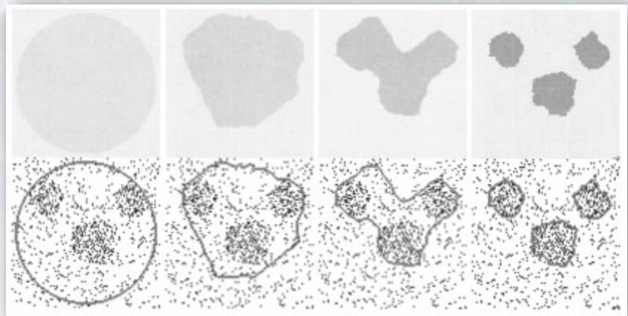
Image processing 1 : filtering and restoration

Lecture 3

Image processing 2 : active contours

goal:

draw one or several contours, around an image or a set of points (segmentation)



general idea:
1) curve obtained by minimization of a criterion combining smoothness, scale parameters, and a measure of “close to image”
2) use volume integral to define and differentiate these quantities (instead o “variational level set” based on level set function alone)

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scale parameters:

length : $\mathcal{L}(\phi) = \int_\Omega |\nabla\phi| \frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})dx$

area inside the curve : $\mathcal{A}(\phi) = \int_\Omega H(\phi)dx$ or $\mathcal{A}(\phi) = \int_\Omega H_\epsilon(\phi)dx$

smoothness:

curvature : $\mathcal{C}(\phi) = \int_\Omega G(\kappa(\phi))|\nabla\phi| \frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})dx$

edge detector: $\mathcal{D}(\phi) = \int_\Omega g(|\nabla u_0(x)|)\, |\nabla\phi| \frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\, dx$

where $g(z) \to 0$ when $z \to \infty$

Typical functional to minimize:

$$\mathcal{E}(\phi) = \lambda\mathcal{L}(\phi) + \alpha\mathcal{A}(\phi) + \gamma\mathcal{C}(\phi) - \delta\mathcal{D}(\phi)$$

with positive coefficients.

Gradient computations:

$$< \mathcal{L}'(\phi), \psi >= -\int_\Omega \frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\kappa(\phi)\,\psi\,dx$$

$$< \mathcal{D}'(\phi), \psi >= -\int_\Omega \left[g(\nabla u_0)\kappa(\phi) + \nabla\left(g(\nabla u_0)\right)\cdot\mathbf{n}\right]\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\,\psi\,dx$$

$$< \mathcal{C}'(\phi)\,,\,\psi >= \int_\Omega \operatorname{div}\left[-G(\kappa(\phi))\frac{\nabla\phi}{|\nabla\phi|} + \frac{1}{|\nabla\phi|}\mathbb{P}_{\nabla\phi^\perp}\left(\nabla[|\nabla\phi|G'(\kappa(\phi))]\right)\right]\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\,\psi\,dx$$

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Gradient descent method can be set as a curvature motion for Φ :

In case one only keeps length and edge detector:

$$\phi_t = \left\{[\lambda + \delta g(\nabla u_0)]\kappa(\phi) + \delta\nabla\left(g(\nabla u_0)\right)\cdot\mathbf{n}\right\}\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})$$

Remarks:

- curvature motion + motion along detector gradients
- “non morphological” form (but comes from intrinsic curves energies)

Other segmentation tools:

when image gradients are not well behaved, try to recover contours inside and outside which the average signal will fit the original image.

replace the edge detector by the following energy

$$\gamma_1 \int |u_0 - C_1|^2 H_\epsilon(\phi)\, dx + \gamma_2 \int |u_0 - C_2|^2 (1 - H_\epsilon(\phi))\, dx$$

where

$$C_1 = \frac{\int u_0(x) H(\phi)\, dx}{\int H(\phi)\, dx} \quad C_2 = \frac{\int u_0(x)(1 - H(\phi))\, dx}{\int (1 - H(\phi))}\, dx$$

leads to the level set equation (with only length control) :

$$\phi_t = \left\{ \lambda\kappa(\phi) - \gamma_1 (u_0 - C_1)^2 + \gamma_2 (u_0 - C_2)^2 \right\} \frac{1}{\varepsilon} \zeta(\frac{\phi}{\varepsilon})$$

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Extension to multiple phases or multiple channels:

using n level sets functions allow to determine 2^n possible states

two level set functions and four phases

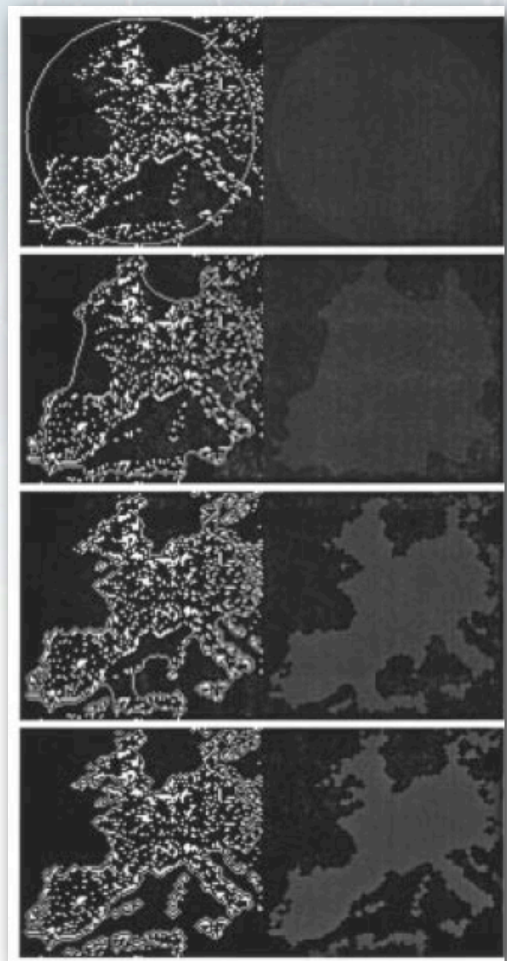
$$\frac{\partial \phi_1}{\partial t} = \delta_\varepsilon(\phi_1)\Big\{\mu \mathrm{div}\Big(\frac{\nabla \phi_1}{|\nabla \phi_1|}\Big) - \Big[\Big((u_0 - c_{11})^2 - (u_0 - c_{01})^2\Big)H(\phi_2) + \Big((u_0 - c_{10})^2 - (u_0 - c_{00})^2\Big)(1 - H(\phi_2))\Big]\Big\},$$

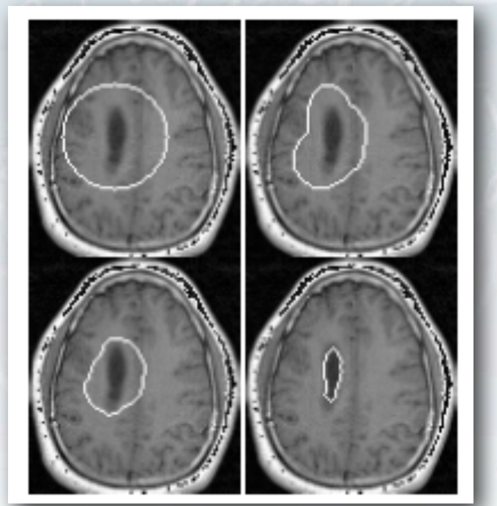
and

$$\frac{\partial \phi_2}{\partial t} = \delta_\varepsilon(\phi_2)\Big\{\mu \mathrm{div}\Big(\frac{\nabla \phi_2}{|\nabla \phi_2|}\Big) - \Big[\Big((u_0 - c_{11})^2 - (u_0 - c_{01})^2\Big)H(\phi_1) + \Big((u_0 - c_{10})^2 - (u_0 - c_{00})^2\Big)(1 - H(\phi_1))\Big]\Big\}.$$

illustrations:

segmentation of
a set of points

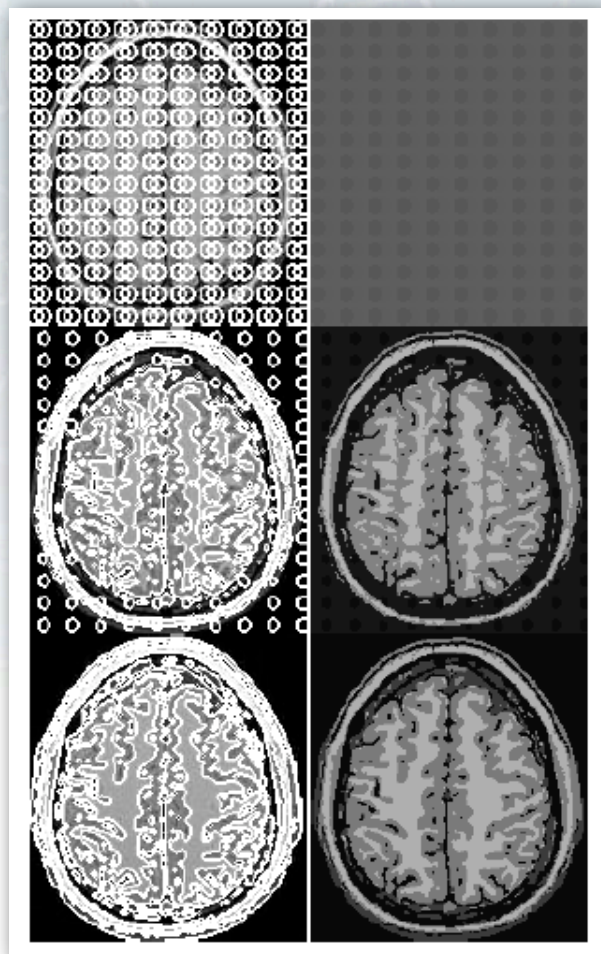




Brain MRI: 2 phases

Brain segmentation:
2 level sets and four phases

delicate point : parameter choice



Lectures 3 and 4: Level set methods for fluid-structure interaction

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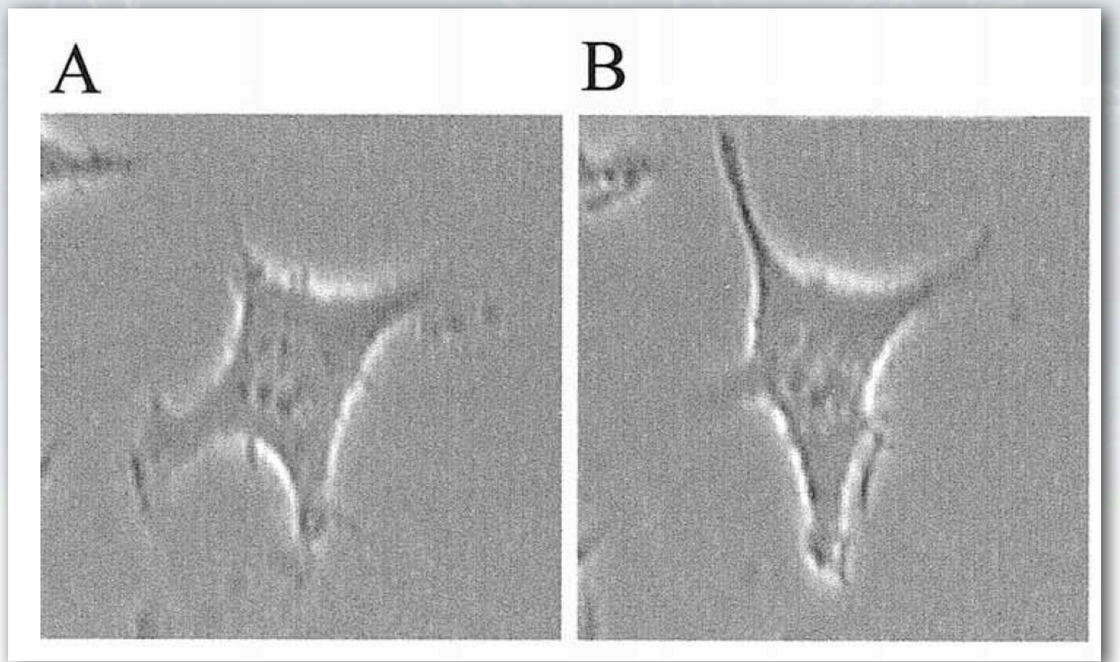
Motivation

Understand spontaneous deformations (or subject to external fields: chemiotaxis) and migration of cell populations



Example of spontaneous oscillations in a fibroblast

What are the relevant parameters (fluid mechanics, rheology, and biochemical processes) to explain such dynamics

Possible approaches for fluid-membrane coupling

- ALE Methods:

  Principle: follow geometry, mesh domain and solve flow and structure equations with appropriate interface conditions

  ✓pro: accurate, explicit enforcement of interface conditions

  ✓con: expensive, difficult to follow large deformations (3D)

- Immersed boundary techniques:

  Principle: flow everywhere, membrane as a forcing term

  ✓con: accuracy questionable (possible leaking)

  ✓pro: cheap, large deformations OK

Immersed boundary techniques (Peskin '77)

Physical setting: elastic fibers moving freely in a fluid, imparting an elastic force on fluid particles

Model:

- Fluid equations, with forcing term localized on fibers (Dirac mass supported by a curve)
- Fibers are tracked as Lagrangian elements
- Elastic forces obtained by explicit computation of stretching along fibers, with elastic coefficient evolving in time to account for contractile activity

Fiber (or membrane) tracked as a Lagrangian element:

$\mathbf{X}(r,s,t)$, r being a, index (scalar for curve, vector for surfaces) parameter, s is parameter along the fiber

and $d\mathbf{X}/dt=\mathbf{u}(\mathbf{X},t)$

Elastic force appears as a right hand side in flow equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right)+\boldsymbol{\nabla}p-\nu\Delta\mathbf{u} = \mathbf{f}$$

where

$$\mathbf{f}(\mathbf{x},t)=\int \mathbf{F}(r,s,t)\delta(\mathbf{x}-\mathbf{X}(r,s,t))drds$$

and similar formula for ρ

$\mathbf{F}$ and M are respectively the force and surface mass densities

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Elastic force density can be derived from energy principle:

$$E(\boldsymbol{\gamma}) = \int_0^1 \mathcal{E}(\|\boldsymbol{\gamma}_s(s)\|)\, ds$$

where $\boldsymbol{\gamma}_s = \dfrac{\partial \mathbf{X}}{\partial s}$ denotes the stretching of the fiber

$$dE(\boldsymbol{\gamma})(\boldsymbol{\delta}) = \int_0^1 \mathcal{E}'(\|\boldsymbol{\gamma}_s(s)\|)\frac{\boldsymbol{\gamma}_s \cdot \boldsymbol{\delta}_s}{\|\boldsymbol{\gamma}_s(s)\|}\, ds$$

$$\mathbf{F} = -\frac{\partial}{\partial s}\left(\mathcal{E}'(\|\boldsymbol{\gamma}_s(s)\|)\frac{\boldsymbol{\gamma}_s}{\|\boldsymbol{\gamma}_s(s)\|}\right) =: -\frac{\partial}{\partial s}(\mathbf{T}\boldsymbol{\tau})$$

Linear elasticity: $\mathcal{E}'(r) = \lambda(r-1)$

yields $\mathbf{T}(\boldsymbol{\gamma}(s,t),t) = \lambda(\|\boldsymbol{\gamma}_s(s,t)\| - 1)$

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Practical implementation:

- Spread force on nearby grid points (interpolation)
- Solve flow equations
- Push markers on the fibers and compute new stretching and new force

Delicate point: need to insert/delete markers to avoid leak or clustering while maintaining volume and mass; dealing with membranes not clear

Alternative view-point: level set formulation

Use of level set techniques based on three remarks:

•Level set functions can capture Lagrangian interfaces
•More physics can be put in the level set function -> immersed boundary
•Can use directly the level set function (without “cosmetics*”) in the forcing term, by renormalizing the cut-off

*: cosmetics = reintialization

Key remark:

Level set functions can be used to compute stretching (-> elastic force) and more:

In 2D, if ϕ solution to the advection equation, then $\mathcal{L} = \nabla \times \phi$ satisfies

$$\frac{\partial \mathcal{L}}{\partial t} + (u \cdot \nabla)\mathcal{L} = (\mathcal{L} \cdot \nabla)u$$

That is, the equation satisfied by stretching.

Consequence $\|\nabla \phi\|$ gives the amount of membrane stretching

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Extension to 3D case

In 3D surface stretching is given by $\dfrac{\partial \boldsymbol{X}}{\partial \xi_1} \times \dfrac{\partial \boldsymbol{X}}{\partial \xi_2}$

where $(\xi_1, \xi_2) \rightarrow \boldsymbol{X}(t; \xi_1, \xi_2)$ is a parameterization of the surface

But, if ϕ is a level set function, $\nabla\phi$ satisfies the same equation

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"Proof"

$$(\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2})_t = \quad [\mathbf{Du}]\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2} + \mathbf{X}_{\xi_1} \times [\mathbf{Du}]\mathbf{X}_{\xi_2}$$

but

$$(\boldsymbol{\nabla}\phi)_t + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\phi + [D\mathbf{u}]^t \boldsymbol{\nabla}\phi = 0$$

IF A has a vanishing trace (because div **u** = 0):

$$\mathbf{Aa} \times \mathbf{b} + \mathbf{a} \times \mathbf{Ab} = -\mathbf{A}^t(\mathbf{a} \times \mathbf{b})$$

Conclusion

$$\frac{\partial \boldsymbol{X}}{\partial \xi_1} \times \frac{\partial \boldsymbol{X}}{\partial \xi_2}(t,\xi) = \lambda(\xi)\nabla\varphi\big(\boldsymbol{X}(t,\xi),t\big)$$

(in the sequel $\lambda=1$)

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Another way to put it

Soit $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ *de classe* $\mathcal{C}^1$ *et* φ *solution* $\mathcal{C}^1$ *de* $\varphi_t + u \cdot \nabla\varphi = 0$*,* $\varphi = \varphi_0$ *avec* $|\nabla\varphi| \geq \alpha > 0$ *au voisinage de* $\{\varphi = 0\}$*. Alors pour toute fonction* f *continue à support compact,*

$$\int_{\{\varphi_0(\xi)=0\}} f(\xi)|\nabla\varphi_0|^{-1}(\xi)d\sigma(\xi) = \int_{\{\varphi(x,t)=0\}} f(Y(x,t))J(x,t)|\nabla\varphi|^{-1}(x,t)d\sigma(x)$$

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Membrane energy can thus be written as

$$\boxed{\mathcal{E}(\phi) = \int_\Omega E(|\nabla\phi|)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})dx}$$

Derivation of force from energy

$$\begin{aligned}
< \mathcal{E}'(\phi)\,,\,\psi >= &\int_\Omega E'(|\nabla\phi|)\frac{\nabla\phi\cdot\nabla\delta}{|\nabla\phi|}\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon}) + E(|\nabla\phi|)\frac{1}{\varepsilon^2}\zeta'(\frac{\phi}{\varepsilon})\psi\,dx \\
&= -\int_\Omega \mathrm{div}\left[E'(|\nabla\phi|)\frac{\nabla\phi}{|\nabla\phi|}\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\right]\psi - E(|\nabla\phi|)\frac{1}{\varepsilon^2}\zeta'(\frac{\phi}{\varepsilon})\psi\ dx \\
&= -\int_\Omega \mathrm{div}\left[E'(|\nabla\phi|)\frac{\nabla\phi}{|\nabla\phi|}\right]\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\psi + (E'(|\nabla\phi|)|\nabla\phi| - E(|\nabla\phi|))\frac{1}{\varepsilon^2}\zeta'(\frac{\phi}{\varepsilon})\psi\,dx
\end{aligned}$$

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Energy time variation:

$$\frac{d}{dt}\mathcal{E}(\phi) =< \mathcal{E}'(\phi)\,,\, \phi_t >=< \mathcal{E}'(\phi)\,,\, -u\cdot\nabla\phi >$$

$$\frac{d}{dt}\mathcal{E}(\phi) = \int_\Omega \mathrm{div}\left[E'(|\nabla\phi|)\frac{\nabla\phi}{|\nabla\phi|}\right]\frac{\nabla\phi}{|\nabla\phi|}|\nabla\phi|\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})u + (E'(|\nabla\phi|)|\nabla\phi| - E(|\nabla\phi|))\frac{1}{\varepsilon^2}\zeta'(\frac{\phi}{\varepsilon})\nabla\phi\cdot u\,dx$$

Integration by parts + div u =0 :

$$\begin{aligned}
&\int_\Omega (E'(|\nabla\phi|)|\nabla\phi| - E(|\nabla\phi|))\nabla\left[\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\right]\cdot u dx \\
&\quad = \int_\Omega (E'(|\nabla\phi|)|\nabla\phi| - E(|\nabla\phi|))\,\mathrm{div}\left[\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})u\right]dx \\
&\quad = -\int_\Omega (\nabla[E'(|\nabla\phi|)]|\nabla\phi| + E'(|\nabla\phi|)\nabla|\nabla\phi| - \nabla[E(|\nabla\phi|)])\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})u dx \\
&\quad = -\int_\Omega \nabla[E'(|\nabla\phi|)]|\nabla\phi|\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})u\,dx
\end{aligned}$$

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should equal the work of the force :

$$\frac{d}{dt}\mathcal{E}(\phi) = -\int_\Omega F(x,t)\cdot u dx$$

by identification of terms:

$$F(x,t) = \left\{\nabla[E'(|\nabla\phi|)] - \mathrm{div}\left[E'(|\nabla\phi|)\frac{\nabla\phi}{|\nabla\phi|}\right]\frac{\nabla\phi}{|\nabla\phi|}\right\}|\nabla\phi|\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})$$

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Other form that distinguishes between tangential and normal components

$$\mathbb{P}_{\nabla\phi^\perp}(v) = v - (v \cdot \frac{\nabla\phi}{|\nabla\phi|})\frac{\nabla\phi}{|\nabla\phi|}$$

$$F(x,t) = \left\{\mathbb{P}_{\nabla\phi^\perp}\left(\nabla[E'(|\nabla\phi|)]\right) - E'(|\nabla\phi|)\kappa(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right\}|\nabla\phi|\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})$$

Can also be written as a tensor term + gradient term

$$F_m = \nabla\left\{E'(|\nabla\varphi|)|\nabla\varphi|\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\right\} - \operatorname{div}\left(E'(|\nabla\varphi|)\frac{\nabla\varphi\otimes\nabla\varphi}{|\nabla\varphi|}\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\right)$$

so that the full flow stress tensor becomes

$$\boxed{\sigma = -p\mathbb{I} + \eta(\nabla u + \nabla u^t) + E'(|\nabla\varphi|)\frac{\nabla\varphi\otimes\nabla\varphi}{|\nabla\varphi|}\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})}$$

Force can be put in conservative form:

$$F_e\,\delta_{\Gamma_t} = \mathrm{div}\left(\frac{E'(|\nabla\phi|)}{|\nabla\phi|}\nabla\phi\otimes\nabla\phi\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\right) - \nabla\left(E'(|\nabla\phi|)|\nabla\phi|)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\right)$$

Remarkable property: energy conservation

kinetic energy viscous dissipation elastic energy

$$\boxed{\begin{aligned}\frac{1}{2}\int_\Omega u^2(x,T)dx + \nu\int_0^T\int_\Omega \|\nabla u\|^2 dxdt + \int_\Omega E(\|\nabla\phi\|)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})dx \\ = \frac{1}{2}\int_\Omega u_0^2(x)dx + \int_\Omega E(\|\nabla\phi_0\|)\frac{1}{\varepsilon}\zeta(\frac{\phi_0}{\varepsilon})dx\end{aligned}}$$

Consequence: no dissipation induced by the regularization

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Level set model (with surface mass)

$$\rho_\varepsilon(\phi) = \bar\rho + (\rho_f - \bar\rho)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})$$

$$\rho_\varepsilon(\phi)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = \mathbf{F}_\varepsilon(x,t) \quad , \quad \operatorname{div}\mathbf{u} = 0$$

$$\phi_t + \mathbf{u}\cdot\nabla\phi = 0$$

where

$$\mathbf{F}_\varepsilon(x,t) = \mathbf{F}(x,t)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})|\nabla\phi|$$

Remarks:

1. Assumes stretching and level set functions normalized at t=0 and interface ={x, ϕ=0}
2. Mass conservation easily satisfied
3. Regularized force can be derived directly from regularized energy

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Korteweg fluid

Assume E'(r)=λr, force becomes:

$$F_m = \operatorname{div}(\nabla\phi \otimes \nabla\phi \frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon}))$$

Set

$\psi = \sqrt{\varepsilon} Z(\frac{\varphi}{\varepsilon})$ with $Z(r) = \int_0^r \zeta^{\frac{1}{2}}(r)dr$

system becomes:

$$\psi_t + u \cdot \nabla\psi = 0$$
$$u_t + u \cdot \nabla u - \nu\Delta u + \nabla p = -\operatorname{div}(\nabla\psi \otimes \nabla\psi)$$
$$\operatorname{div} u = 0$$

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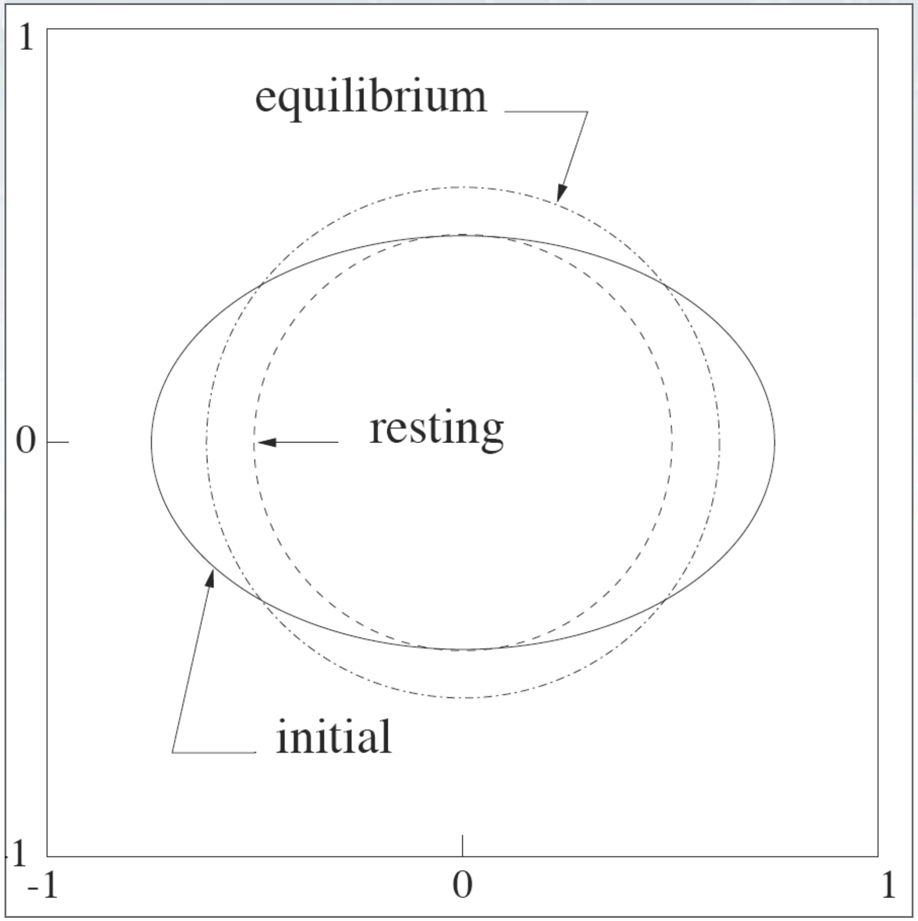
Existence result

Under reasonable assumption on E, and for smooth initial conditions, there exists a smooth solution for small time

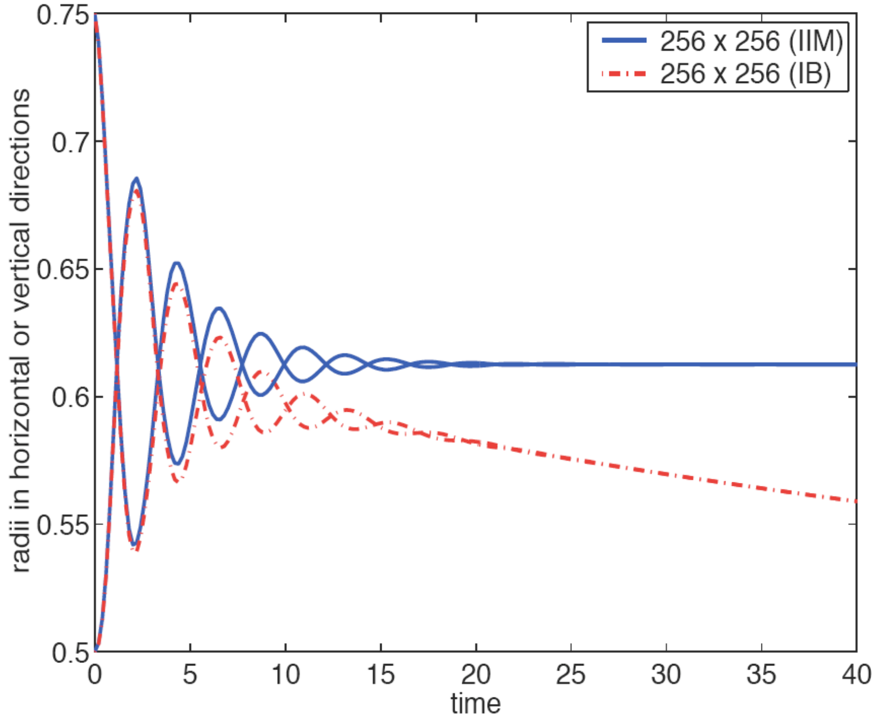
Existence time tends to zero with ϵ !

Idea of the proof:
compactness argument on a sequence of time-delay linearized/regularized problems
Solonikov L^p estimates on Stokes problem

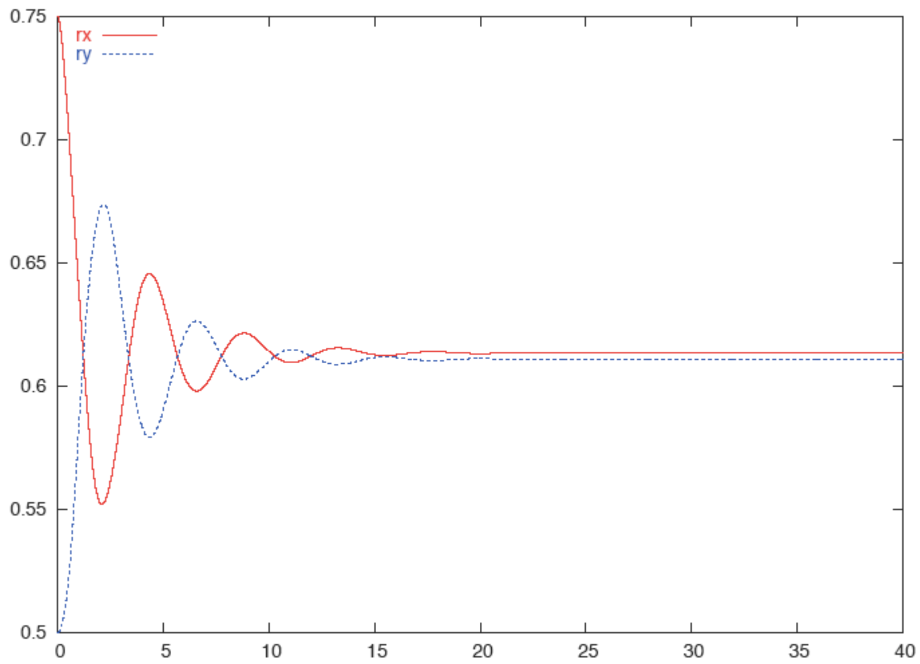
Numerical validation against existing IBM: pressurized membrane (Lee-Levesque, JCP '03)



Radius in horizontal and vertical direction N=256

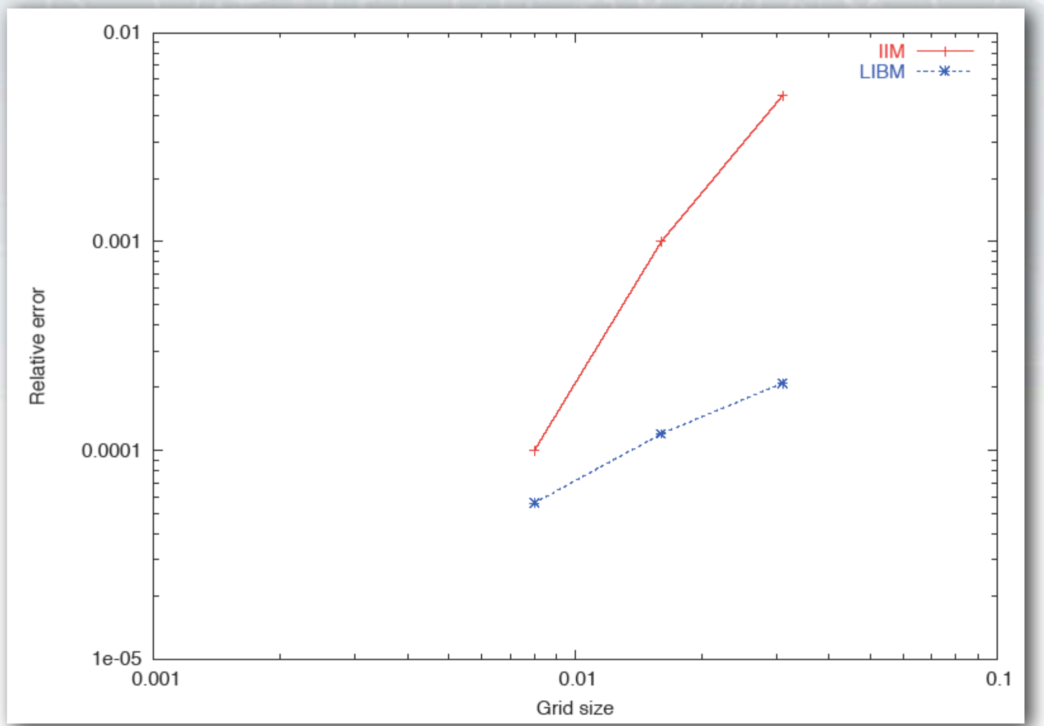


Original and L-L implementation of IBM



Level-set method

Conservation of volume



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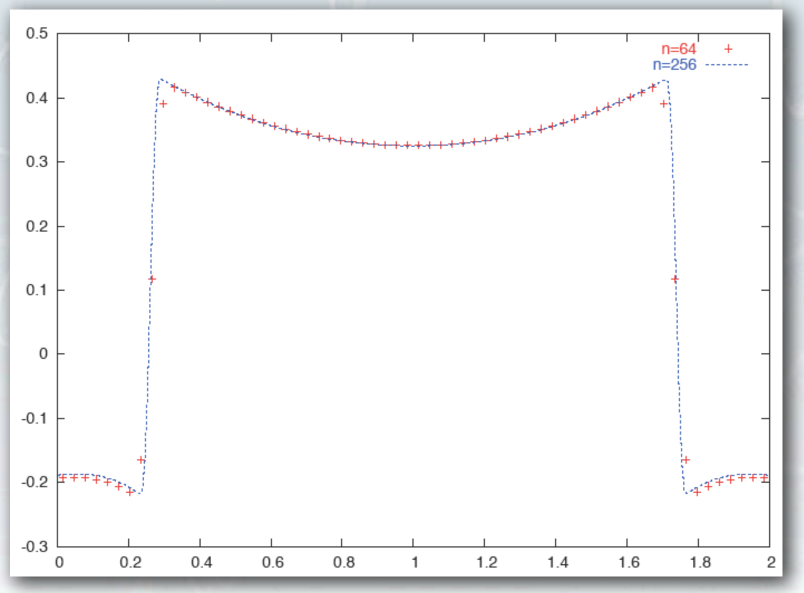

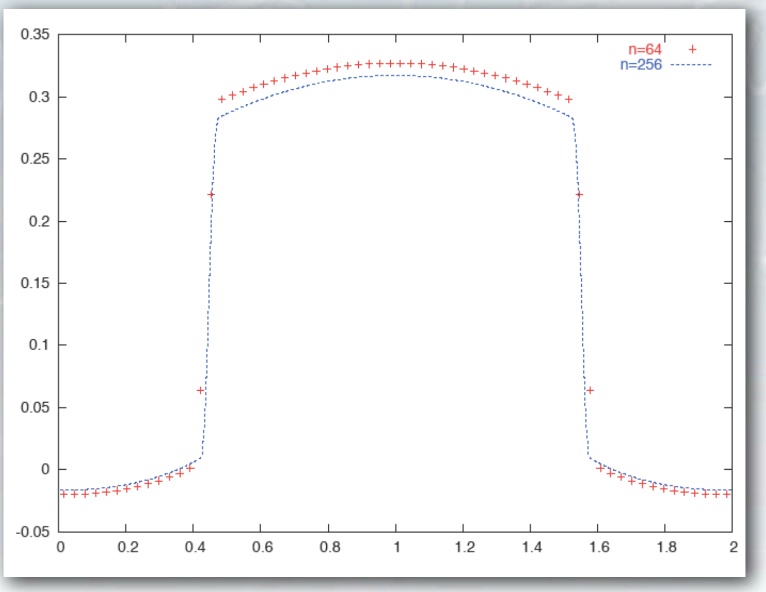

T=0.2

T=2.2

Pressure gradients at x=0 for N=64 and N=256

If you want to play by yourself :

In 2D:

http://ljk.imag.fr/membres/Emmanuel.Maitre/Logiciels

In 3D:

http:/ljk.imag.fr/COMMA/news.html

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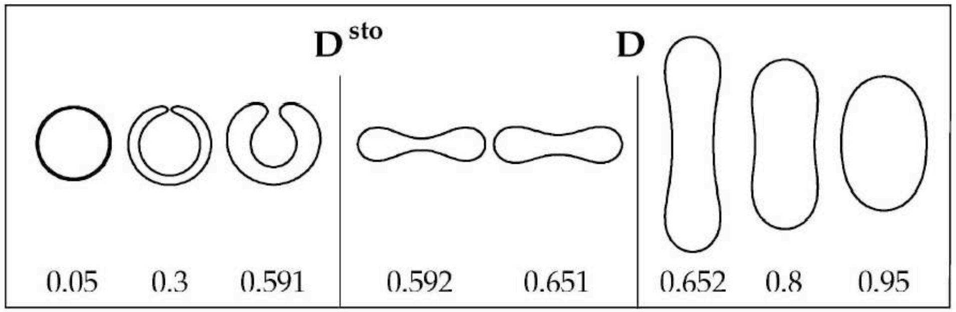
Application : equilibrium shapes for biological vesicles as minimal shapes for curvature energy, with volume and area constraints

function of $$\tau = \frac{V}{\frac{4}{3}\pi\left(\frac{A}{4\pi}\right)^{3/2}} \in [0,1]$$

where V is the volume, A the area.

expected shapes



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Level set approach to shape optimization:

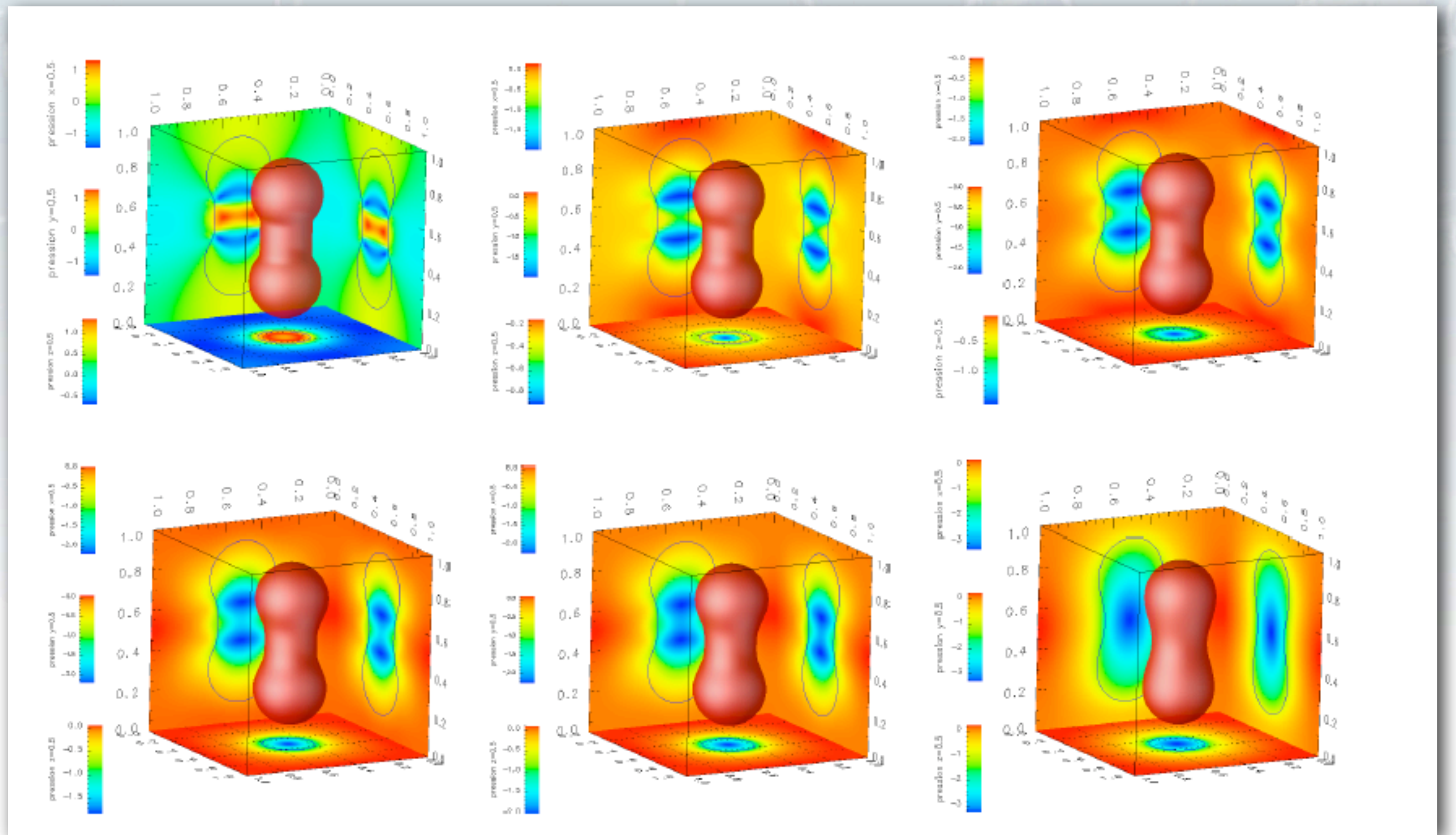
$$E_c(\phi) = \int G(\kappa(\phi))|\nabla\phi|\frac{1}{\epsilon}\zeta\left(\frac{\phi}{\epsilon}\right)$$

$$G(r) = \frac{r^2}{2}\ ,\ \kappa(\phi) = \boldsymbol{\nabla}\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

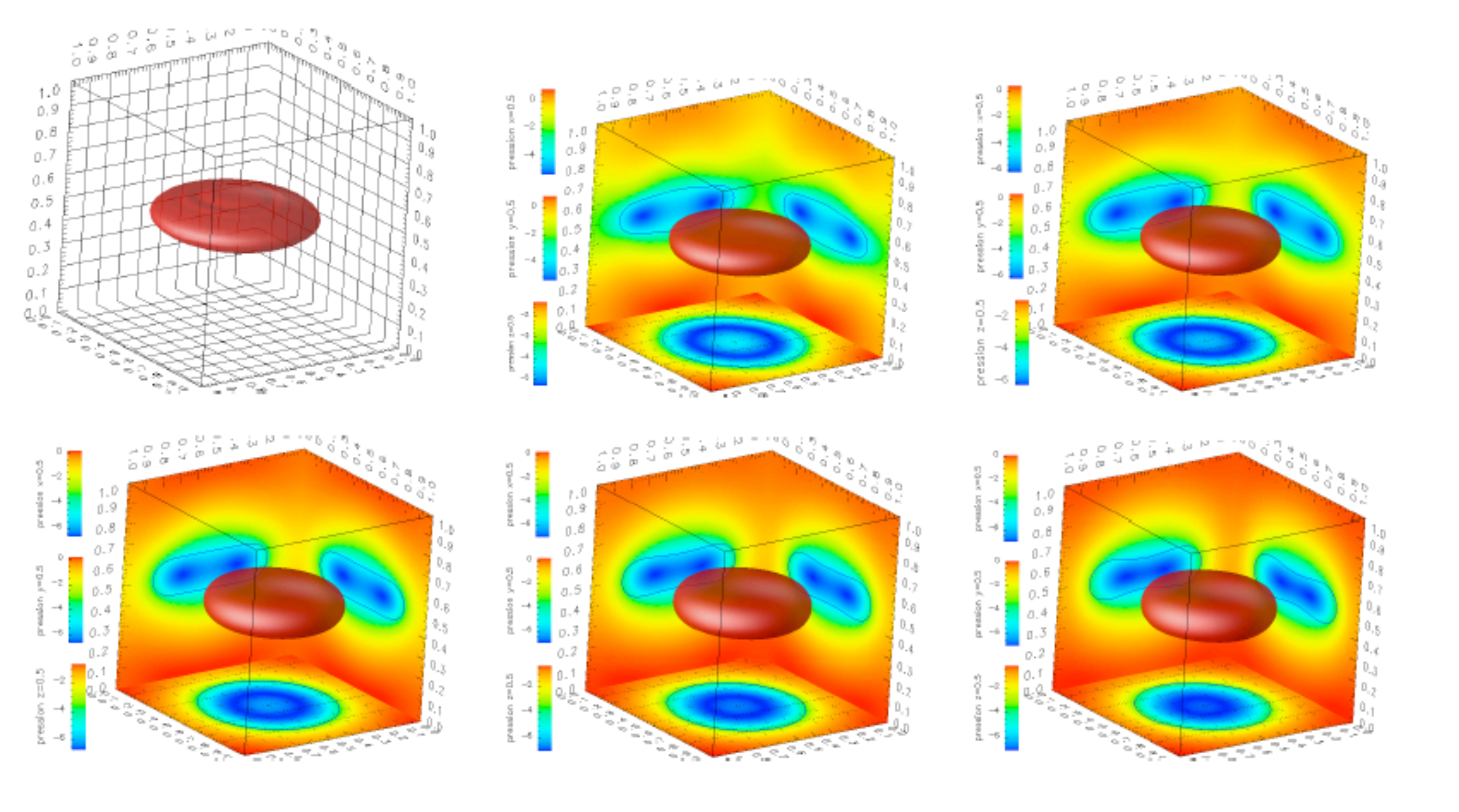
curvature driven energy and flow: $\frac{dE_c}{dt} = -\int \mathbf{F}_c\cdot\mathbf{u}$

$$\mathbf{F}_c = \boldsymbol{\nabla}\cdot\left\{G(\kappa(\phi))\frac{\nabla\phi}{|\nabla\phi|} + \frac{1}{|\nabla\phi|}P_{\nabla\phi^\perp}(\nabla(|\nabla\phi|G'(\kappa(\phi))))\right\}\frac{1}{\epsilon}\zeta(\frac{\phi}{\epsilon})\nabla\phi$$

gradient based method replaced by **incompressible** (to maintain volumes) flow solver with RHS= Fc + **stiff elastic force** (to maintain area)



$\tau=0.8$



τ=0.58

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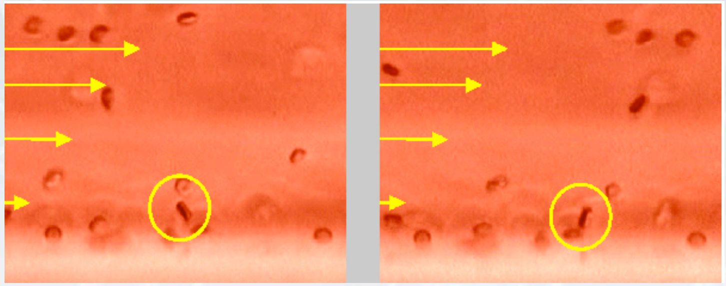
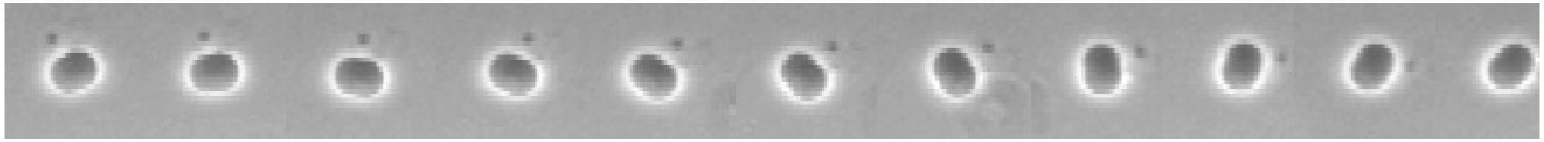
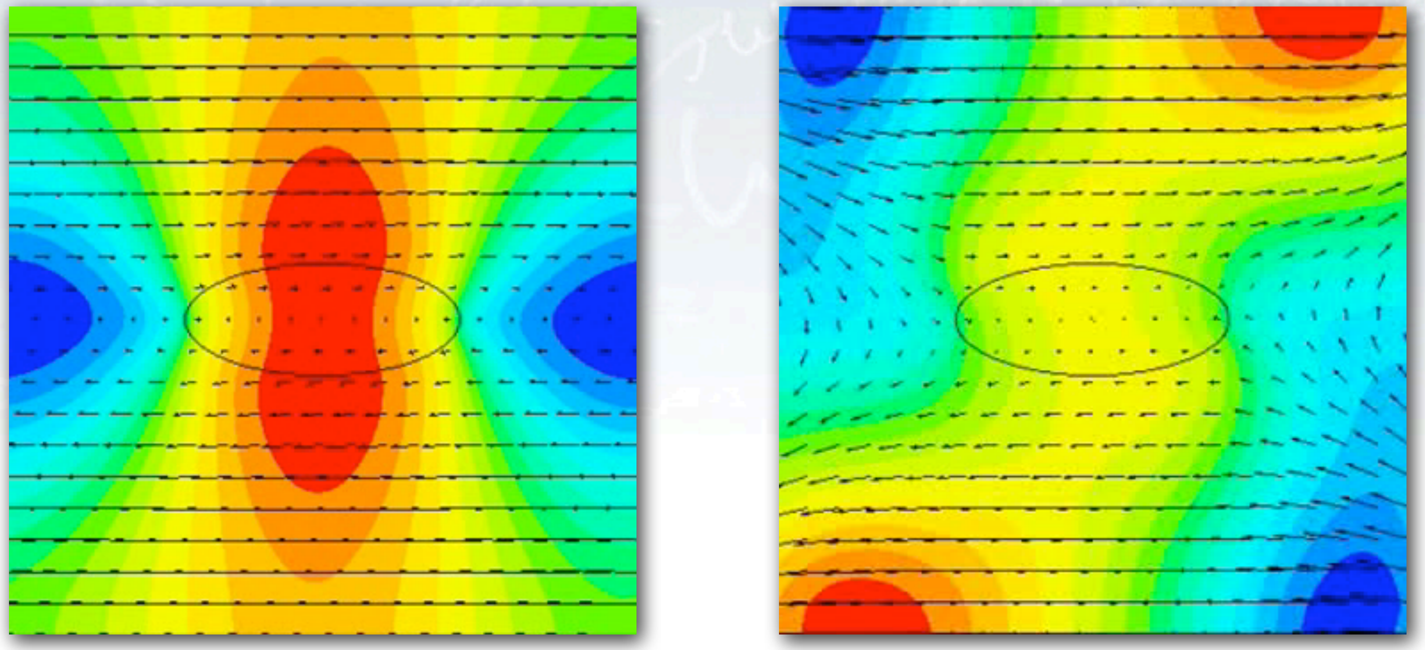
Other application in biological vesicles: tumbling and tank-trading in shear flows , depending on viscosity ratio

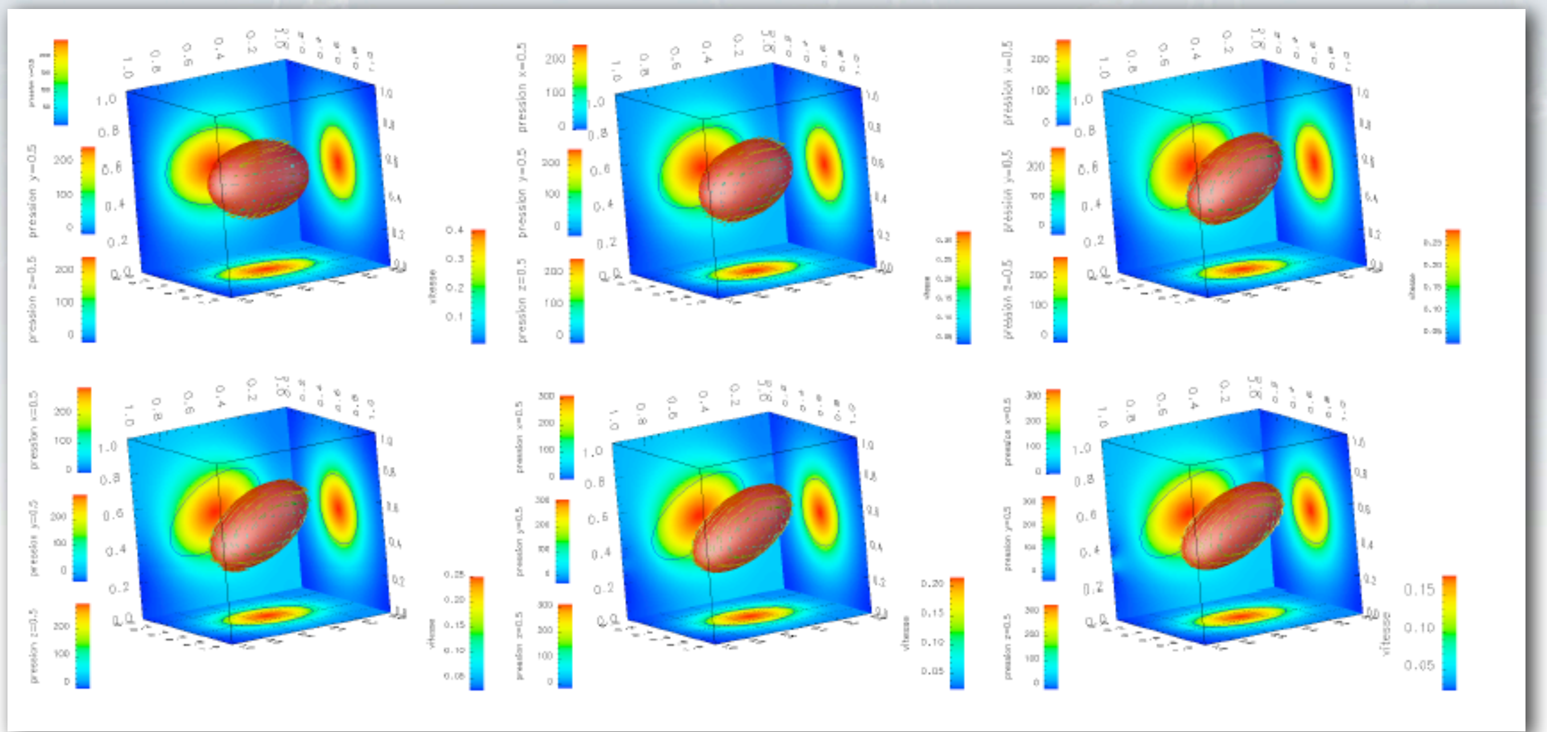




In 2D: tank treading vs tumbling



In 3D :tank treading (viscosity ratio = 1)



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Fluid-structure with general incompressible elastic bodies

conservation of momentum $u_t + u \cdot \nabla u - \mathrm{div}_x \boldsymbol{\sigma} = f$

Consider general strain/stress relationship:

$$\boldsymbol{\sigma}(x) = \boldsymbol{\sigma}^D(\xi, (\nabla X \nabla X^t)(t; \xi, 0))$$

$$\boldsymbol{\sigma}^D(\xi, B) = \beta_0(\xi, \iota_B) I + \beta_1(\xi, \iota_B) B + \beta_2(\xi, \iota_B) B^2$$

where X are the forward characteristics and ξ are the Lagrange coordinates.

If one considers the backward characteristics $(x,t) \to X(0;x,t)$

then $\nabla X(t,\xi,0)\nabla X(0;x,t) = \mathbb{I}_d$

and Cauchy-Green tensor can be written as

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \text{ where } \mathbf{F}(x,t) = (\boldsymbol{\nabla}\mathbf{X})(t;\mathbf{X}(0;\mathbf{x},t),0) = (\boldsymbol{\nabla}\mathbf{X})^{-1}(0;\mathbf{x},t)$$

important remark:

because det F = 1,
$$\mathbf{F} = cof\nabla\mathbf{X}^t = \begin{pmatrix} \mathbf{X}_{,x_2} \times \mathbf{X}_{,x_3} \\ \mathbf{X}_{,x_3} \times \mathbf{X}_{,x_1} \\ \mathbf{X}_{,x_1} \times \mathbf{X}_{,x_2} \end{pmatrix}$$

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why consider backward flow?

1) it is a function of eulerian coordinate

2) it satisfies a transport equation
$$\begin{cases} X_t + u \cdot \nabla X = 0, \\ X = x \end{cases}$$

this allows to write a general level-set formulation of elasticity.

For a fluid-structure system, need one more level-set function ψ to track the interface.

Full model:
$$(\mathbf{u}_t + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}) - \mathrm{div}\ \boldsymbol{\sigma}(\psi, \mathbf{Du}, \mathbf{X}, \boldsymbol{\nabla}\mathbf{X}) + \boldsymbol{\nabla} p = 0$$

where

$$\boldsymbol{\sigma}(\psi, \mathbf{Du}, \mathbf{X}, \boldsymbol{\nabla}\mathbf{X}) = \boldsymbol{\sigma}^S(\mathbf{X}, \boldsymbol{\nabla}\mathbf{X}^t\boldsymbol{\nabla}\mathbf{X}) + H(\psi/\varepsilon)\left(\boldsymbol{\sigma}^F(\mathbf{Du}) - \boldsymbol{\sigma}^S(\mathbf{X}, \boldsymbol{\nabla}\mathbf{X}^t\boldsymbol{\nabla}\mathbf{X})\right)$$

(H is a regularized heavyside function)

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How to take into account anisotropic elasticity ?

Let $C = \nabla_\xi X^t \nabla_\xi X$ the (right) Cauchy-Green tensor and τ a preferred direction.

Constitutive equations should involve 2 additional invariants:

$I_4 = \tau^t C \tau$ (stretching in direction τ) and $I_5 = \tau^t C^2 \tau$ or $I_5 = \tau^t C^{-1} \tau$

To represent τ we introduce an (additional) level set φ_0 and set $\tau = \nabla\varphi_0$

If is the advected level set function we can write, using backward trajectories:

$$I_4 = \nabla\varphi_0 \, \nabla_x X^{-t} \;\; \nabla_x X^{-1} \nabla\varphi_0 = \nabla\varphi \, \nabla_x X^{-1} \nabla_x X^{-t} \;\; \nabla_x X^{-1} \nabla_x X^{-t} \nabla\varphi_0 = | \, B \, \nabla\varphi \, |^2$$

Similarly:

$$I_5 = \; | \, \nabla\varphi \, |^2$$

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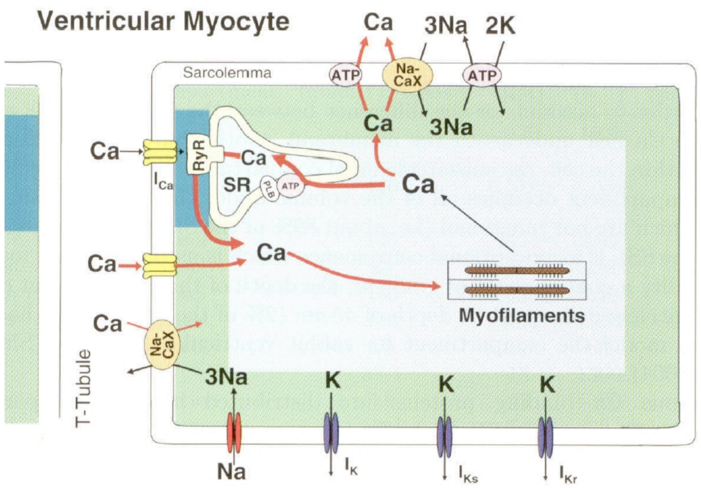
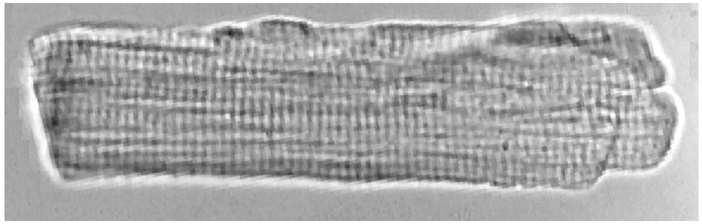
Application: dynamics of a single heart tissue cell (cardiomyocyte)





Gauche : Copyright T.R. Shannon, LNM 1867, Springer p. 65. Droite : Image issue d'une séquence de vidéo-microscopie de cardiomyocyte isolé de rat

coupling of active stress and calcium concentration in an elastic incompressible medium

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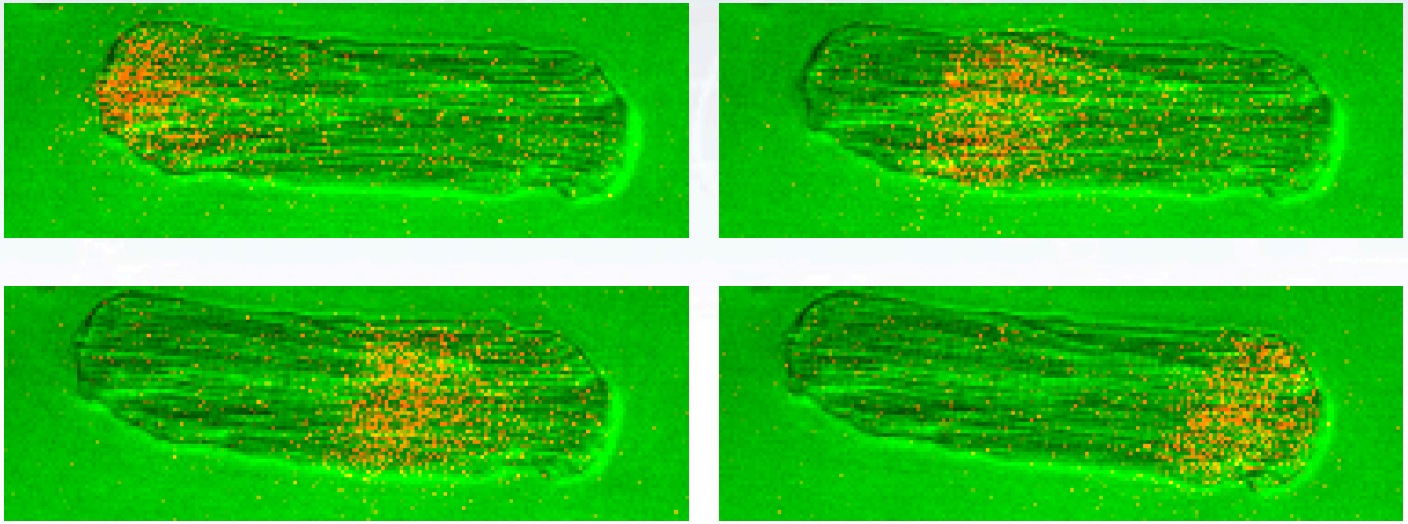
spontaneous contraction is related to propagation of calcium waves:



Experiments by Usson and Lacampagne

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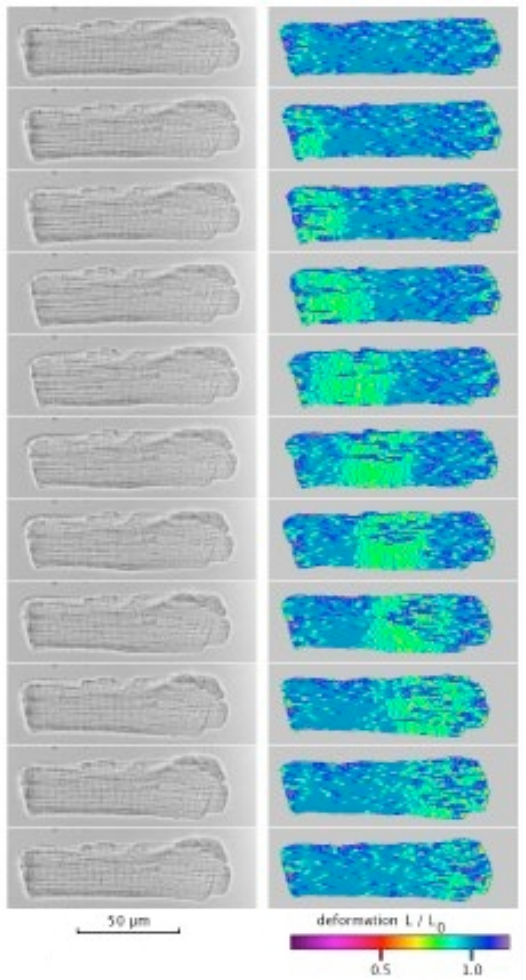
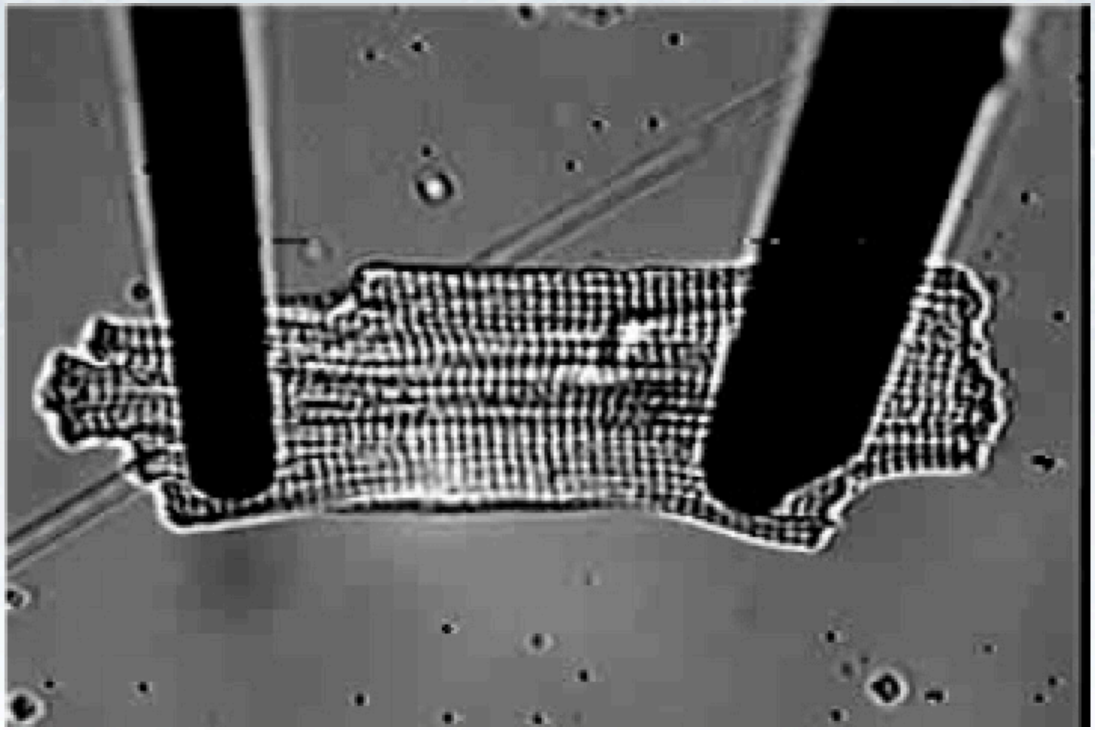
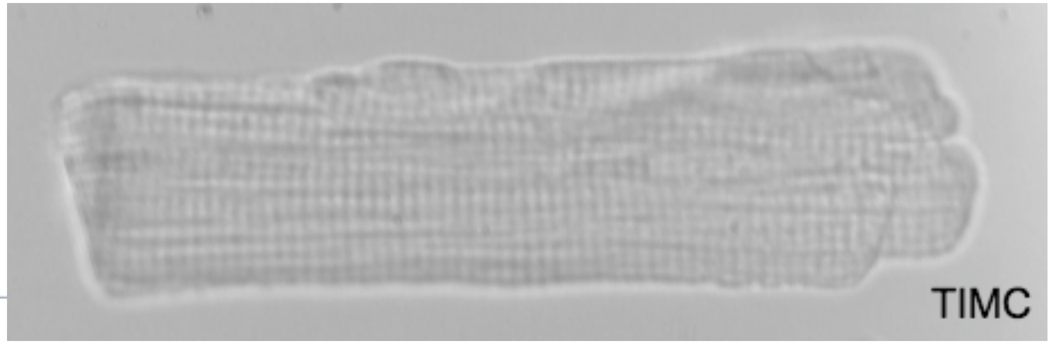




measure of deformation



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mathematical modeling : incompressible elastic medium
with preferred direction along the filaments
(isotropic transverse model (Ogden, Mourad, ..))

results in the following constitutive law:

$$\boldsymbol{\sigma}^S = -p + 2\alpha_1 \mathbf{B} + 2\alpha_2((\mathbf{B})\mathbf{B} - \mathbf{B}^2) + 2\alpha_4 \mathbf{F}\tau \otimes \mathbf{F}\boldsymbol{\tau} + ...$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \text{ where } \mathbf{F}(x,t) = (\boldsymbol{\nabla}\mathbf{X})(t;\mathbf{X}(0;\mathbf{x},t),0) = (\boldsymbol{\nabla}\mathbf{X})^{-1}(0;\mathbf{x},t)$$

coupling with calcium concentration

Assumption (Dupont, Goldbeter, Stephanou, Tracqui):

- calcium concentration acts on σ^S through coefficient α_4
- calcium concentration in the cell Z and in the reticulum Y satisfy a reaction-diffusion model

$$\frac{\partial Y}{\partial t} = \nu_2(Z) - \nu_3(Y,Z) - k_f Y$$
$$\frac{\partial Z}{\partial t} = \nu_0 + \nu_1\beta - \nu_2(Z) + \nu_3(Y,Z) + k_f Y - kZ + \nabla \cdot (\mathbf{D}\nabla Z)$$
$$Z(r,t_0) = Z_0;\ Y(r,t_0) = Y_0$$

$$\nu_2 = V_{M_2}\frac{Z^n}{K_2^n + Z^n} \qquad \nu_3 = V_{M_3}\frac{Y^m}{K_R^m + Y^m}\frac{Z^p}{K_A^p + Z^p}$$

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Parameter	Value	Unit
ν_0	0.45	$\mu M.s^{-1}$
k	2.2	s^{-1}
ν_1	4	$\mu M.s^{-1}$
V_{M_2}	65	$\mu M.s^{-1}$
V_{M_3}	500	$\mu M.s^{-1}$
K_2	1.2	μM
K_A	0.92	μM
K_R	3.5	μM
Y_0	0.1	μM
Z_0	10	μM
Z_{50}	2.5	μM
β	0.05	–
D_{11}	300	$\mu m^2 s^{-1}$
D_{22}	150	$\mu m^2 s^{-1}$
D_{33}	150	$\mu m^2 s^{-1}$
T_0	5.5	kPa

Implementation

- Finite-difference weno scheme
- MAC-type grid
- One cycle of contraction (2s of physical time) on a 100^3 grid takes 3 hours of CPU on AMD 64b processeur
- To be compared to Okada et al : 34 hrs for 1s of physical time

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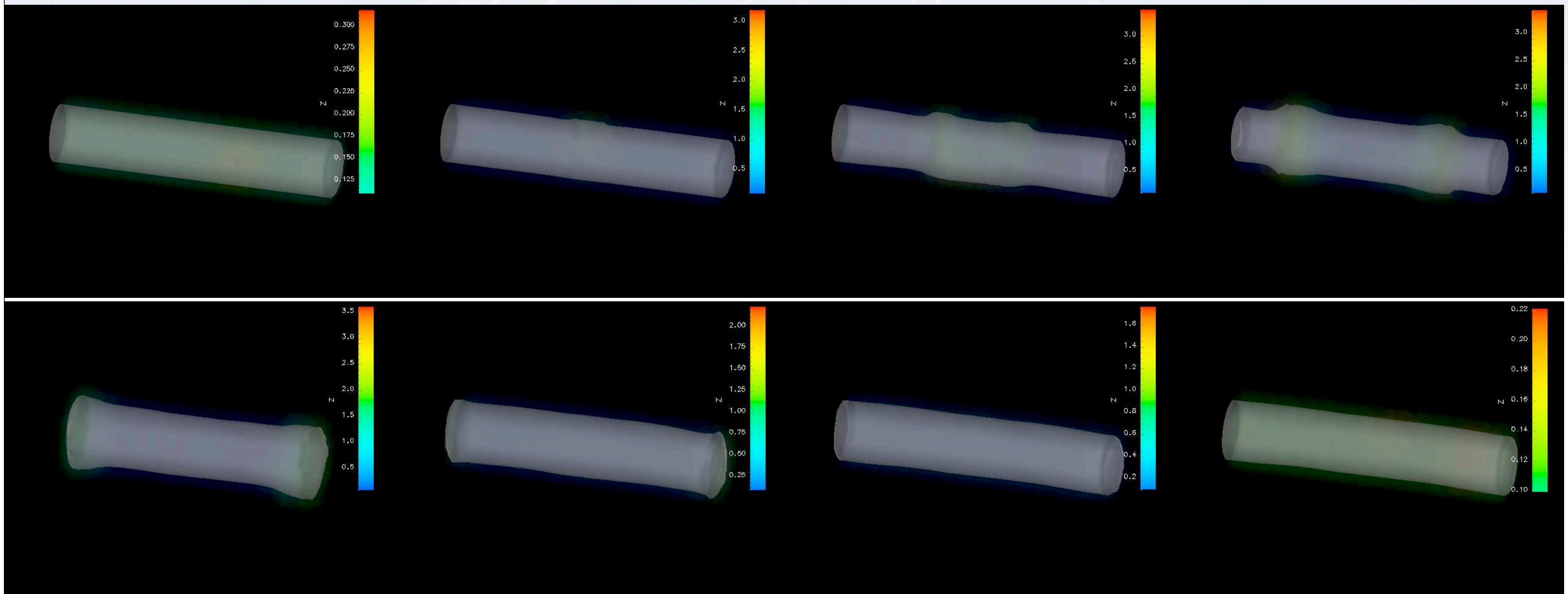
Propagation of deformation in a cylinder
(calculation by E. Maitre)



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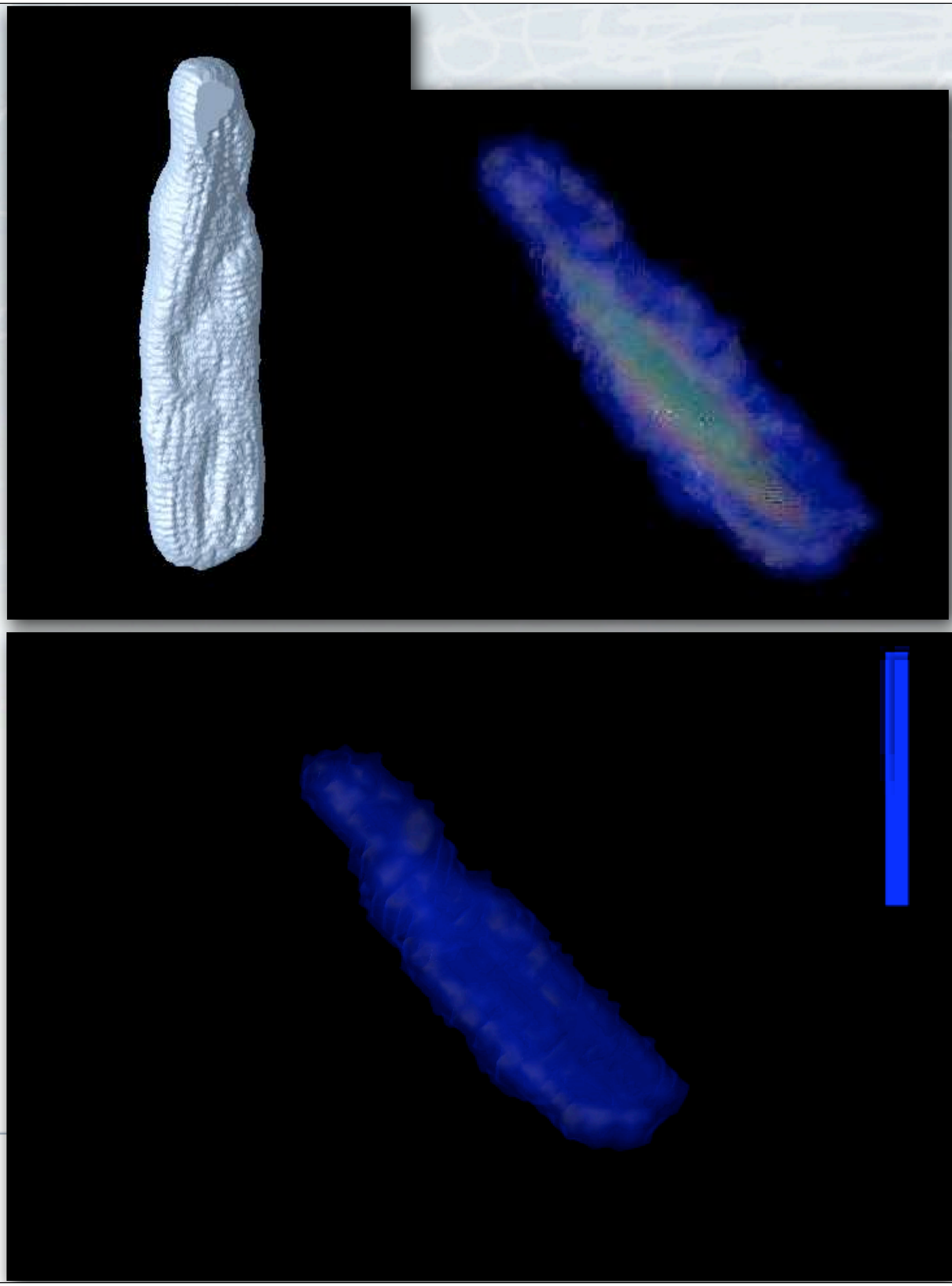
Calcium wave in a cardiomyocyte

(C., Maitre & Milcent ECCOMAS CFD Conf., 2006)



Last lecture:

- how to deal with contact/collision using level set methods
- a few things about stability issues in level set methods for fluid-structure interactions

Usual way to deal with contacts of solid:

- geometry based methods :detect collision or penetration, then “eject” the overlapping
- solid mechanics: translate “no penetration” through a Lagrange multpilier (in ALE variational framework) or assume central repelling forces between objects

dealing with contacts with level set methods

idea:

✓start form 1D dynamical system with following properties

- short range
- energy preserving
- parameter free (as much as possible)

✓then "spread it" on the surface of the bodies using level set functions

1D contact model : $\ddot{x} = \frac{\kappa}{x} \exp(-x/\epsilon)$.

Hamiltonian system with energy $E(x) = \int_1^x \frac{\kappa}{s} \exp(-s/\epsilon), \quad ds = \int_{1/\epsilon}^{x/\epsilon} \frac{\kappa}{y} \exp(-y)\, dy$

Consider a point initially located at x(0)=1 with negative initial velocity v_0 and set

$$F_\epsilon(x) = \int_x^{1/\epsilon} \frac{1}{y} \exp(-y)\, dy, \quad F(x) = \int_x^{+\infty} \frac{1}{y} \exp(-y)\, dy$$

If x^* denotes contact point, by conservation of energy : k $F(x^*/\epsilon) = v_0^2$

$$\boxed{x^\star = \epsilon G_\epsilon(v_0^2/2\kappa) \simeq \epsilon G(v_0^2/2\kappa) \quad \text{for } \epsilon \ll 1}$$

consequence:

1) rebound on a width ~ ϵ

2) κ should be scaled as v^2 where v is the velocity before contact

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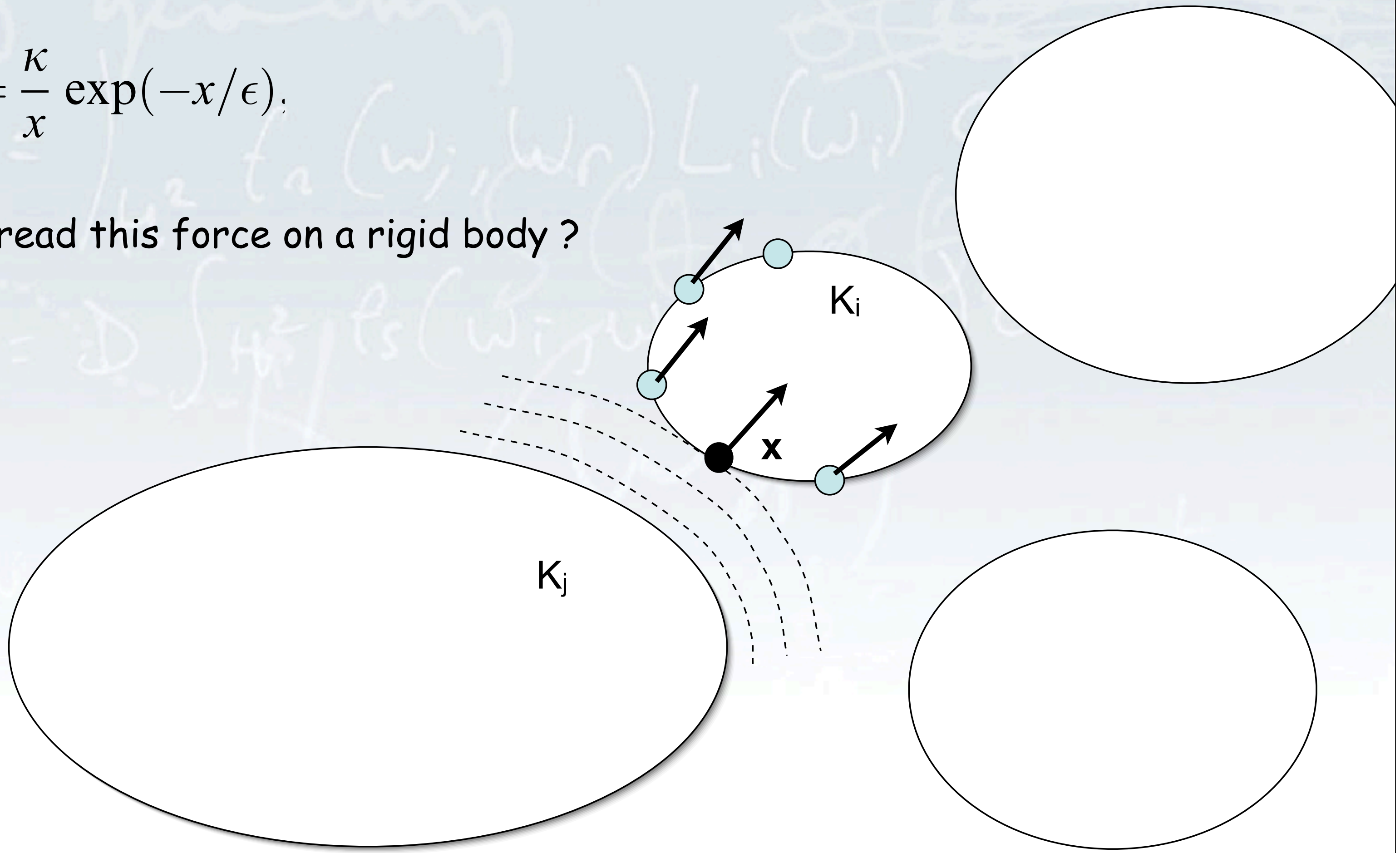
$$\ddot{x} = \frac{\kappa}{x}\exp(-x/\epsilon).$$

How to spread this force on a rigid body ?



$$\mathbf{f}_{\text{col}}(\mathbf{x}) = \sum_{ij} \frac{\nabla\phi_j(\mathbf{x})}{|\nabla\phi_j(\mathbf{x})|} \quad \frac{\kappa_{ij}}{\phi_j(\mathbf{x})}\exp\left(-\phi_j(\mathbf{x})/\epsilon\right) \quad \frac{|\nabla\phi_j(\mathbf{x})|}{\epsilon}\zeta\left(\frac{\phi_i(\mathbf{x})}{\epsilon}\right)$$

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Energy conservation
and contact scale ~ ϵ

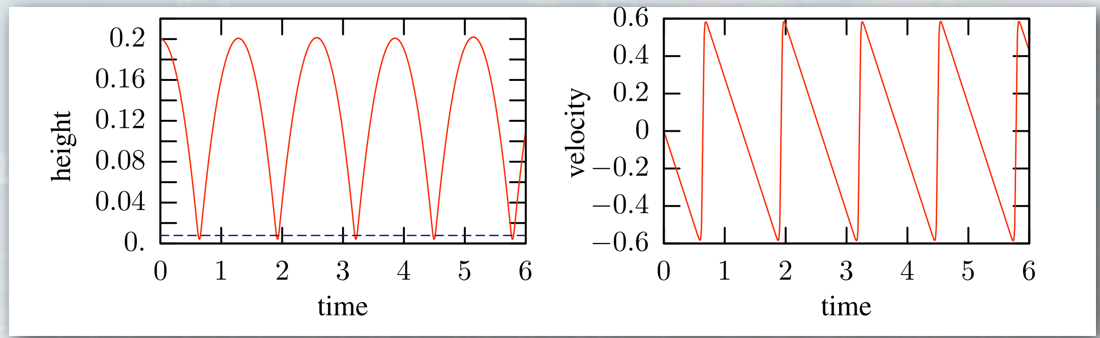
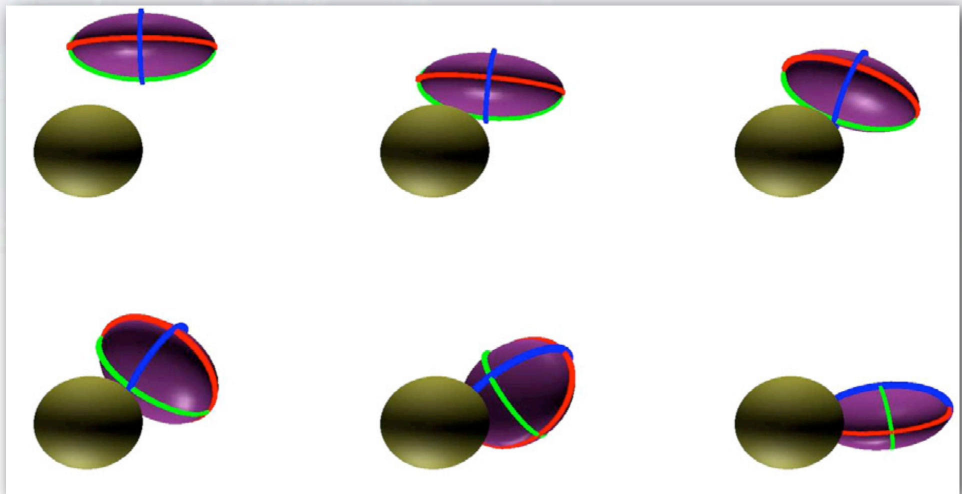


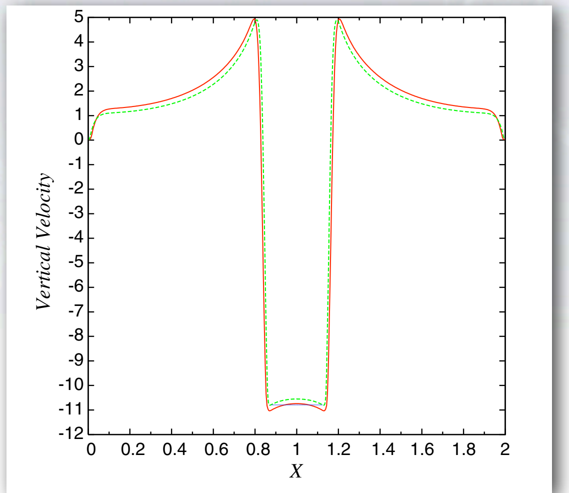
Illustration of contact force



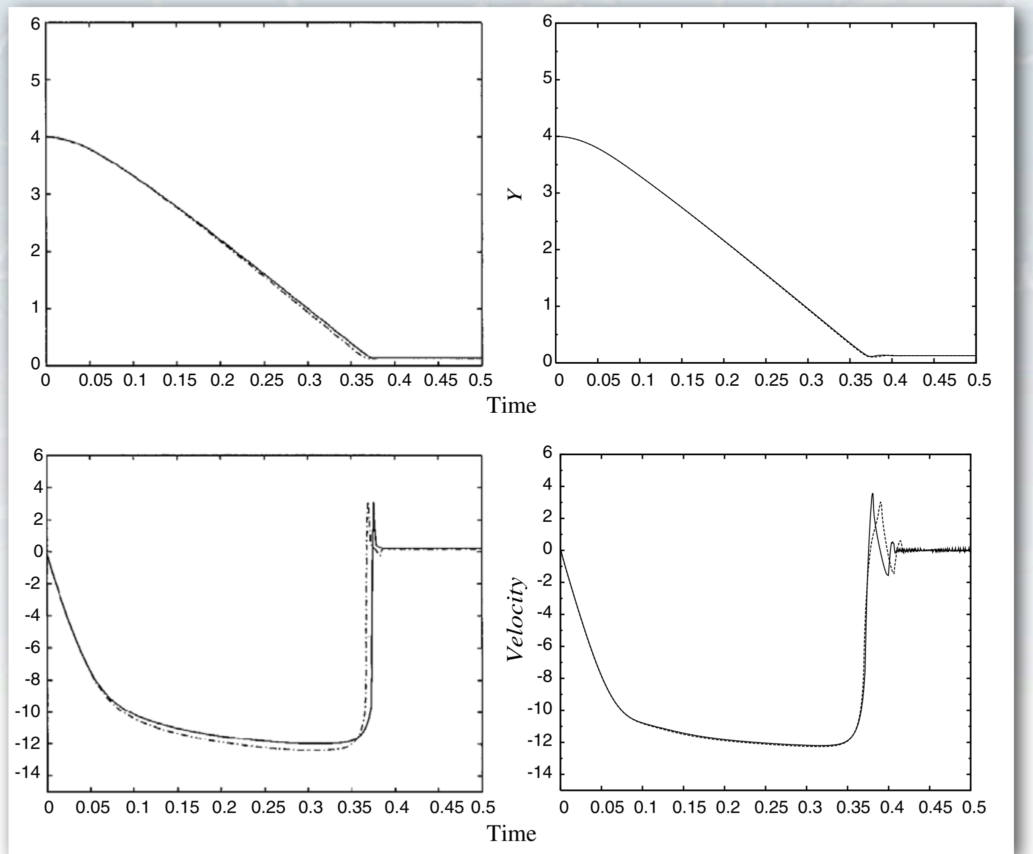
difference with central force:
able to production rotation
if contact "off-center"

disk hitting a wall

2D disk sedimentation



free fall:
vertical velocity for 2 grid sizes



comparison with Glowinski et al. 2001 (left pictures: flow model, Lagrange multiplier for rigidity constraint) : vertical velocity (top) and location (bottom) for 2 grid sizes.

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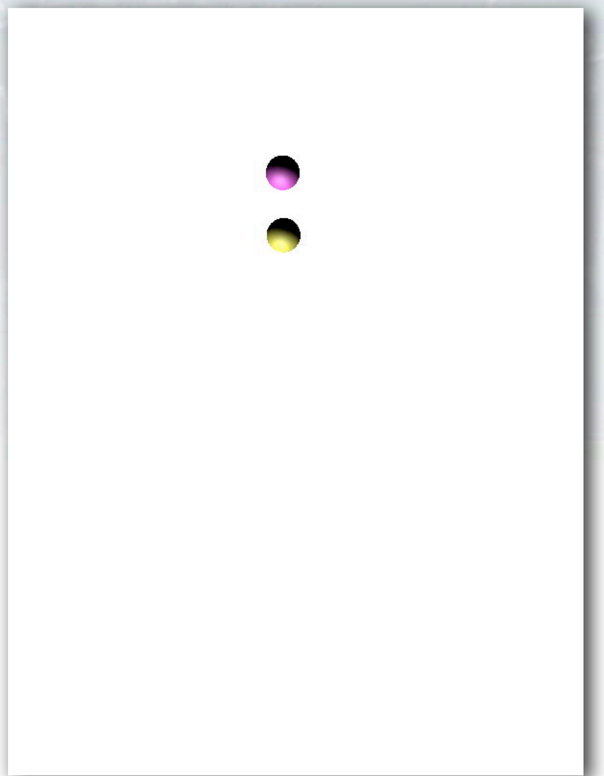
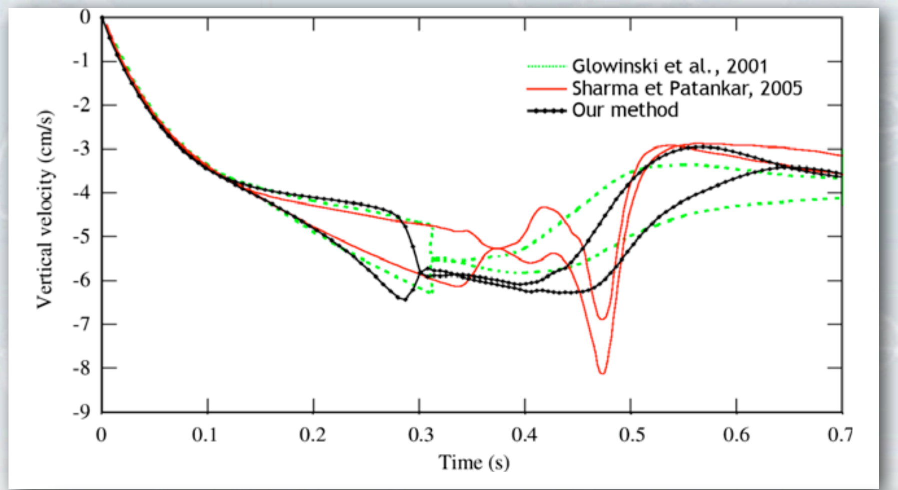
kissing and tumbling of spheres

comparisons with

Patankar and Glowinski

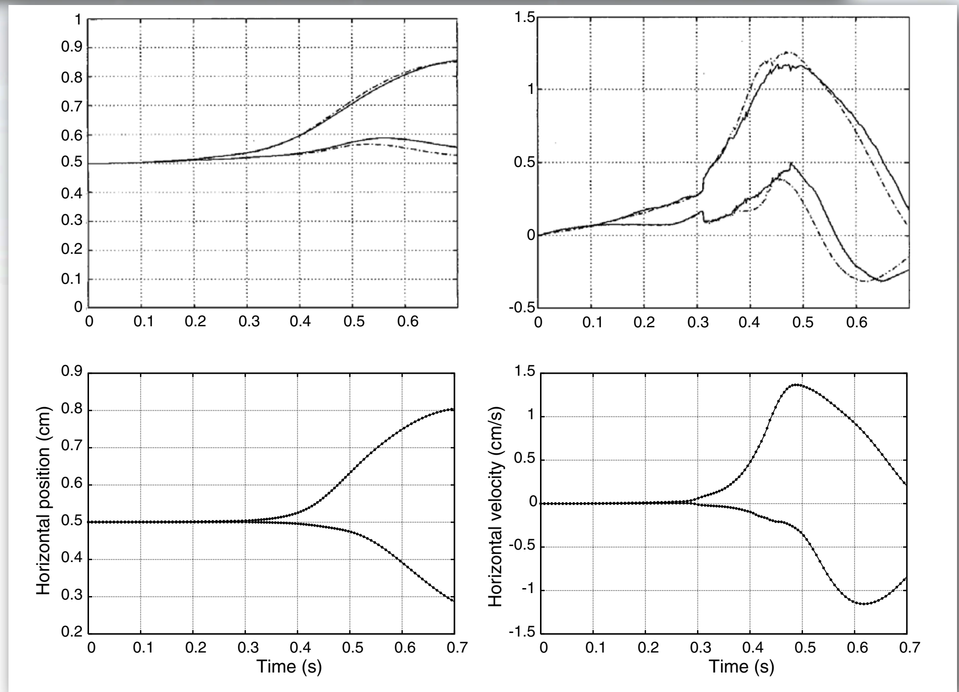
(density ratio=1000)





vertical velocities

horizontal velocity and position →



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Stability issues in fluid-structure interaction

whatever method, difficult problem (mathematically and numerically
in general no clear-cut criteria
explicit methods require small time-steps
fully implicit methods time consuming (poor convergence for fixed point iterations in non-linear part of algorithm)

here we try to give a tentative numerical analysis of the membrane/fluid interaction toy problem (1D !)

Problem under consideration (2D)

$$
\begin{cases}
\rho\left(u_t + u.\nabla u\right) + \nabla p - \mu \Delta u = F_e \\
F_e = \left[\left(\nabla(E_e'(|\nabla\phi|)) \cdot \dfrac{\nabla\times\phi}{|\nabla\phi|}\right)\cdot \dfrac{\nabla\times\phi}{|\nabla\phi|} - E_e'(|\nabla\phi|)\kappa(\phi)\dfrac{\nabla\phi}{|\nabla\phi|}\right]\dfrac{|\nabla\phi|}{\varepsilon}\zeta\left(\dfrac{\phi}{\varepsilon}\right) \\
\operatorname{div} u = 0 \\
\phi_t + u\cdot\nabla\phi = 0
\end{cases}
$$

Particular case $E'(r) = \nu_e$ leads to

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + u.\nabla u - \mu\Delta u + \nabla p = -\nu_e \kappa(\phi)\nabla\phi\dfrac{1}{\varepsilon}\zeta\left(\dfrac{\phi}{\varepsilon}\right), \\
\operatorname{div} u = 0, \\
\dfrac{\partial \phi}{\partial t} + u.\nabla\phi = 0.
\end{cases}
$$

1D version :

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} - \mu\dfrac{\partial^2 u}{\partial x^2} = -\nu_e\dfrac{\partial^2\phi}{\partial x^2}\dfrac{\partial\phi}{\partial x}\dfrac{1}{\varepsilon}\zeta\left(\dfrac{\phi}{\varepsilon}\right), \\
\dfrac{\partial\phi}{\partial t} + u\dfrac{\partial\phi}{\partial x} = 0.
\end{cases}
$$

Linearized problem around $\begin{cases} \overline{u}(x) = 0, \\ \overline{\phi}(x) = x. \end{cases}$

$$(u, \phi) = (\overline{u} + \widetilde{u}, \overline{\phi} + \widetilde{\phi})$$

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} + \widetilde{u}\dfrac{\partial \widetilde{u}}{\partial x} - \mu \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = -\nu_e \dfrac{\partial^2 \widetilde{\phi}}{\partial x^2} \left(1 + \dfrac{\partial \widetilde{\phi}}{\partial x}\right) \dfrac{1}{\varepsilon} \zeta \left(\dfrac{\overline{\phi} + \widetilde{\phi}}{\varepsilon}\right), \\ \dfrac{\partial \widetilde{\phi}}{\partial t} + \tilde{u}\dfrac{\partial \widetilde{\phi}}{\partial x} + \tilde{u} = 0. \end{cases}$$

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} - \mu \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = -\nu_e \dfrac{\partial^2 \widetilde{\phi}}{\partial x^2} \dfrac{1}{\varepsilon} \zeta \left(\dfrac{\overline{\phi}}{\varepsilon}\right) + o(\widetilde{u}, \widetilde{\phi}), \\ \dfrac{\partial \widetilde{\phi}}{\partial t} + \widetilde{u} = o(\widetilde{u}, \widetilde{\phi}). \end{cases}$$

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Additional assumption: focus on level set 0, and assume

$$\begin{cases}\zeta(r) = 1 \text{ if } |r| \leq \dfrac{1}{2}, \\ \zeta(r) = 0 \text{ else}\end{cases}$$

leads to

$$\begin{cases}\dfrac{\partial u}{\partial t} - \mu \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\nu_e}{\varepsilon}\dfrac{\partial^2 \phi}{\partial x^2} \text{ on } [0,T] \times \mathbb{R} \\ \dfrac{\partial \phi}{\partial t} + u = 0 \text{ on } [0,T] \times \mathbb{R} \\ u(0,x) = f(x),\ \phi(0,x) = g(x) \text{ on } \mathbb{R}\end{cases}$$

Remark: can rewrite as

$$u = -\frac{\partial \phi}{\partial t}$$

$$\frac{\partial^2 \phi}{\partial t^2} - \mu \frac{\partial^3 \phi}{\partial t \partial x^2} = \frac{\nu_e}{\varepsilon}\frac{\partial^2 \phi}{\partial x^2}$$

(wave equation)

3 natural time-stepping schemes

fully explicit (FE)

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} - \mu\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2} = -\dfrac{\nu_e}{\varepsilon}\dfrac{\phi_{j+1}^n-2\phi_j^n+\phi_{j-1}^n}{(\Delta x)^2} \\ \dfrac{\phi_j^{n+1}-\phi_j^n}{\Delta t} + u_j^n = 0 \\ u_j^0 = f_j,\ \phi_j^0 = g_j \end{cases}$$

semi-implicit (SI)

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} - \mu\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2} = -\dfrac{\nu_e}{\varepsilon}\dfrac{\phi_{j+1}^n-2\phi_j^n+\phi_{j-1}^n}{(\Delta x)^2} \\ \dfrac{\phi_j^{n+1}-\phi_j^n}{\Delta t} + u_j^{n+1} = 0 \\ u_j^0 = f_j,\ \phi_j^0 = g_j \end{cases}$$

fully implicit (FI)

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} - \mu\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2} = -\dfrac{\nu_e}{\varepsilon}\dfrac{\phi_{j+1}^{n+1}-2\phi_j^{n+1}+\phi_{j-1}^{n+1}}{(\Delta x)^2} \\ \dfrac{\phi_j^{n+1}-\phi_j^n}{\Delta t} + u_j^{n+1} = 0 \\ u_j^0 = f_j,\ \phi_j^0 = g_j \end{cases}$$

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Result:

FI is unconditionally stable

FE is stable under condition $\Delta t \leq \frac{\mu\varepsilon}{\nu_e}$

SI is stable under condition $\Delta t < \frac{(\mu\varepsilon + \max(\mu\varepsilon, \sqrt{\nu_e\varepsilon}\Delta x))}{\nu_e}$

Remark: SI better than FE except if large viscosity

For zero viscosity, SI reduces to

$$\begin{cases} \dfrac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{(\Delta t)^2} - \dfrac{\nu_e}{\varepsilon}\dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{(\Delta x)^2} = 0 \\ \phi_j^0 = g_j,\ \phi_j^1 = g_j - \Delta t f_j \end{cases}$$

and stability condition $\Delta t < \frac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}}$ as expected for wave equation

In practice $\epsilon \approx \Delta x$ so the conditions become:

(FE) $$\boxed{\Delta t \leq \frac{\mu \Delta x}{\nu_e}}$$

(SI) $$\boxed{\Delta t < \frac{\mu \Delta x + \max(\mu, \sqrt{\nu_e \Delta x}) \Delta x}{\nu_e}}$$

For small viscosity, get for (SI)

$$\Delta t < \sqrt{\frac{\Delta x^3}{\nu_e}}$$

condition often seen (but not proved) for curvature motion (surface tension)

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Numerical validations: 2D version of (SI)

$$
\begin{cases}
\dfrac{u^{n+1}-u^n}{\Delta t} + u^n.\nabla u^n - \mu \Delta u^{n+1} + \nabla p^{n+1} - F(\phi^n) = 0 \\
F(\phi^n) = \Bigg[\left(E''(|\nabla \phi^n|) \dfrac{\nabla^2 \phi^n \nabla \phi^n}{|\nabla \phi^n|} \dfrac{\nabla \times \phi^n}{|\nabla \phi^n|} \right) \dfrac{\nabla \times \phi^n}{|\nabla \phi^n|} \\
-E'(|\nabla \phi^n|) \kappa(\phi^n) \nabla \phi^n \Big] \dfrac{1}{\varepsilon} \zeta \left(\dfrac{\phi^n}{\varepsilon |\nabla \phi^n|} \right) \\
\operatorname{div} u^{n+1} = 0 \\
\dfrac{\phi^{n+1}-\phi^n}{\Delta t} + u^{n+1} \nabla \phi^n = 0
\end{cases}
$$

SI scheme with $\Delta t \leq \Delta t_A, \quad \Delta t_A = C \dfrac{(\mu + \max(\mu, \sqrt{\nu_e \Delta x}))\Delta x}{\nu_e}$

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Increasing stiffness : compare time step limitation of (SI) and (FE) and curvature motion time step

ν_e	Δt_A with $C = 0.2$	Δt_V with $C = 0.2$	Δt_B
1	1.25×10^{-2}	6.25×10^{-3}	$C \times 5.52427 \times 10^{-3}$
10^2	1.72985×10^{-4}	6.25×10^{-5}	$C \times 5.52427 \times 10^{-4}$
10^4	1.16735×10^{-5}	6.25×10^{-7}	$C \times 5.52427 \times 10^{-5}$
10^6	1.1111×10^{-6}	6.25×10^{-9}	$C \times 5.52427 \times 10^{-6}$

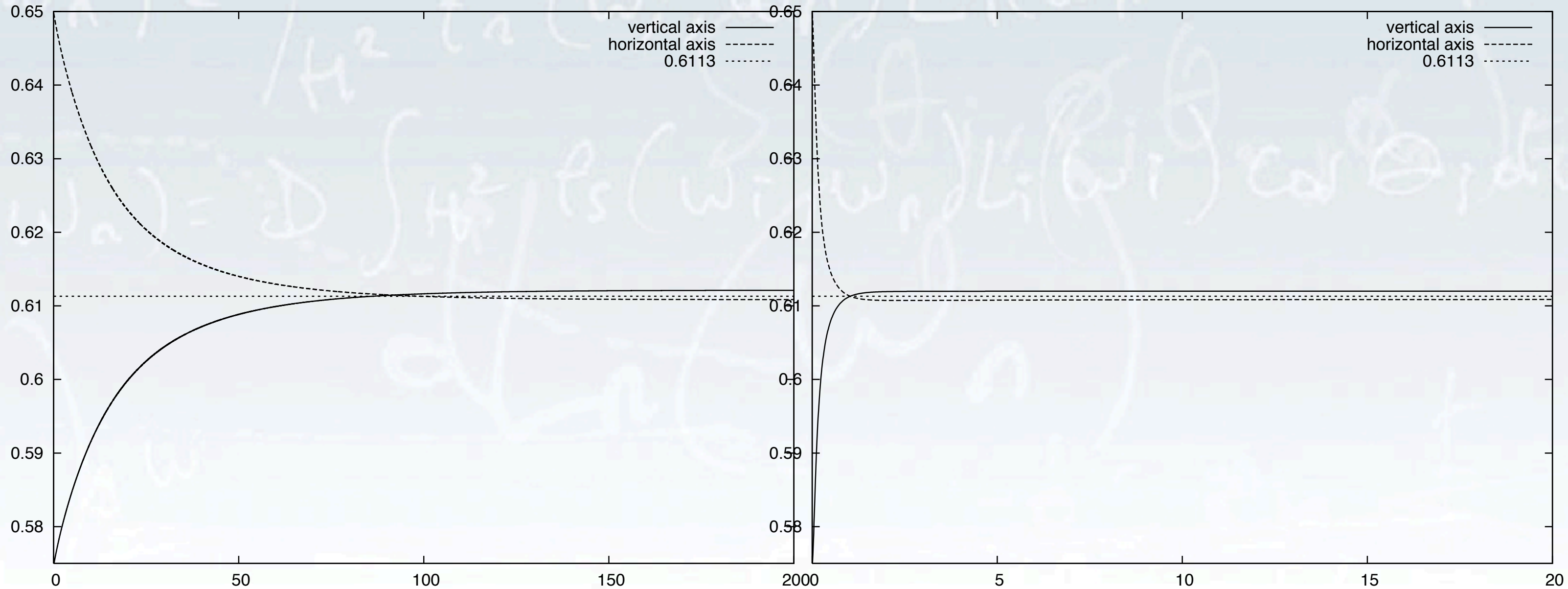

Figure 1: The time evolution of the membrane axes for $\nu_e = 1$ (left) and $\nu_e = 10^2$ (right), $\mu = 1$, $M = 64$.

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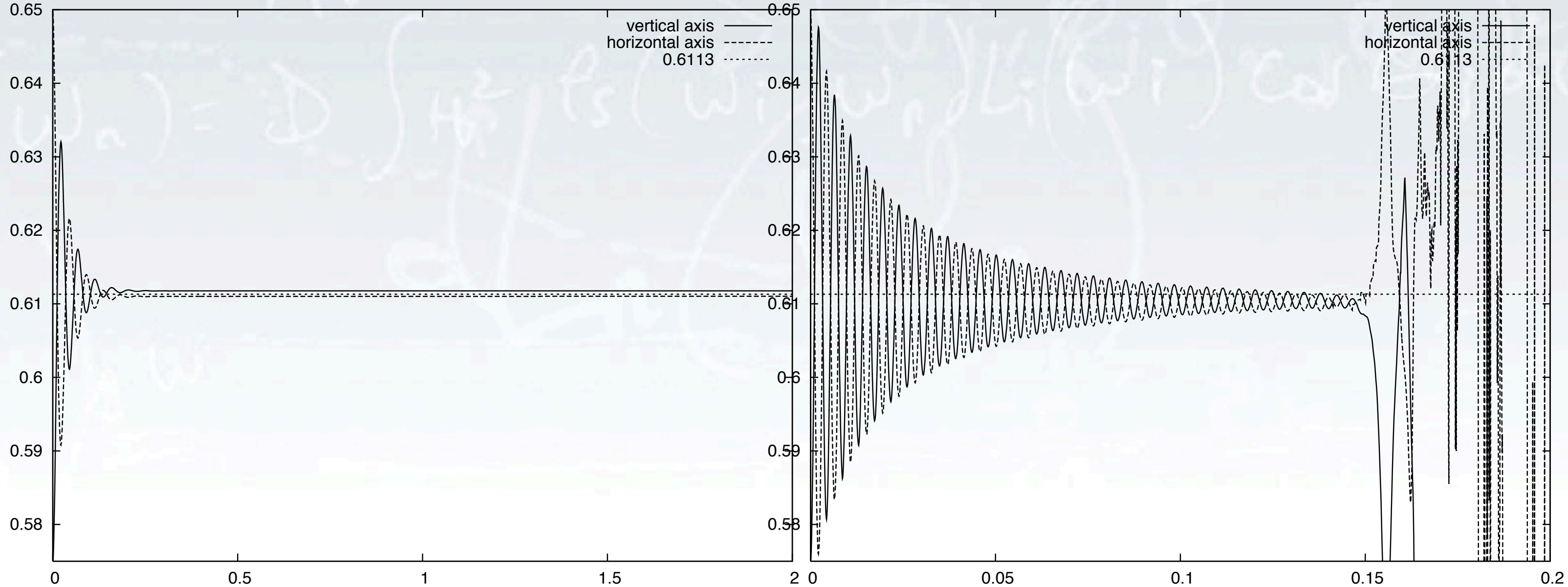

Figure 2: The time evolution of the membrane axes for $\nu_e = 10^4$ (left) and $\nu_e = 10^6$ (right), $\mu = 1$, $M = 64$.

Time steps for various grid sizes, μ=ν=1

M	Δt_A with $C = 0.2$	Δt_V with $C = 0.2$	Δt_B
64	1.25×10^{-2}	6.25×10^{-3}	$C \times 1.10485 \times 10^{-3}$
128	6.25×10^{-3}	3.125×10^{-3}	$C \times 3.90625 \times 10^{-4}$
256	3.125×10^{-3}	1.5625×10^{-3}	$C \times 1.38107 \times 10^{-4}$
	SI	FE	curvature motion

Open problem: define an unconditionnally stable scheme based on FI discretization of linearized (wave equation like) model