Guaranteed, locally efficient, asymptotically exact, and asymptotically robust a posteriori error estimation in the finite element method

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The theory of a posteriori error estimation in numerical approximation of partial differential equations aims at controlling the error between the numerical approximation and the (unknown) exact solution. The ultimate goal is quadruple:

- give an upper bound on the error in the numerical solution, which only uses the approximate solution and can thus be evaluated (guaranteed upper bound);
- give an expression for the estimated error locally, for example in each element of the computational mesh, and assure that this estimate on the error represents a lower bound for the actual error, up to a multiplicative constant (local efficiency);
- assure that the ratio of the estimated error and the actual error goes to one (asymptotic exactness);
- guarantee the three previous properties independently of the data or mesh properties (robustness).

It is really amazing that one may be able to give a bound on the error without the knowledge of the exact solution and it is indeed possible. Three main branches of a posteriori estimates in the finite element method have evolved during the last decades, see the survey books by Ainsworth and Oden [1] and Verfürth [5]. Explicit residual-based estimators (see [5]) are build upon the fact that approximate solution does not satisfy the given partial differential equation, are computationally inexpensive, and usually fulfill the first and second desirable property. However, up to very rare exceptions, the first property is not really satisfied in the strict sense, since one has a computable upper bound up to an unknown multiplicative constant (one usually knows that this constant only depends on e.g. mesh or domain properties and it is always independent of the unknown solution and mesh size); we shall term such estimator as reliable (but not guaranteed). As such, it should rather be called error indicator instead of error estimator; it is completely sufficient for the usual practice, where one only uses it in order to refine mesh elements with increased error, but not for the actual control of the error. Note that in particular, studying the third desirable property loses sense in this case. Implicit residual-based estimators like the equilibrated residual method (see [1]) remove much of the above drawbacks under the condition that local infinite-dimensional problems can be solved. This is of course hardly doable in practice and hence one has to approximate the solutions of these problems, leading to the loss of the guaranteed upper bound and increased computational cost. Finally, averaging-based estimators as the celebrated Zienkiewicz–Zhu one, see [8], are easy to compute, often fulfill the third property, but systematically fail the first one in the strict (guaranteed) sense, although they may actually be shown equivalent with the explicit residual-based ones, cf.

Verfürth [5], thus yielding reliable and efficient estimates (up to higher-order terms), see also [2].

We have recently introduced in [6, 7] computationally inexpensive estimators which fulfill all the four desirable properties (the robustness may however only be asymptotic), in the framework of mixed finite element and finite volume methods. The purpose of this project is to adapt these techniques to the finite element case. At first, a model second-order elliptic problem will be considered, featuring however discontinuous and anisotropic coefficients. Extensions to (singularly-perturbed) convection–diffusion–reaction problems are intended later. In a theoretical part, rigorous analysis of the estimator properties will be done. Implementation and extensive numerical simulations will form an important second part the project, with the purpose to verify the properties of the developed estimator and compare it to some well-known ones, like the Zienkiewicz–Zhu, explicit and implicit residual, or those introduced recently in [3, 4].

References

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