
Imaging in random media

Chrysoula Tsogka

tsogka@math.uchicago.edu

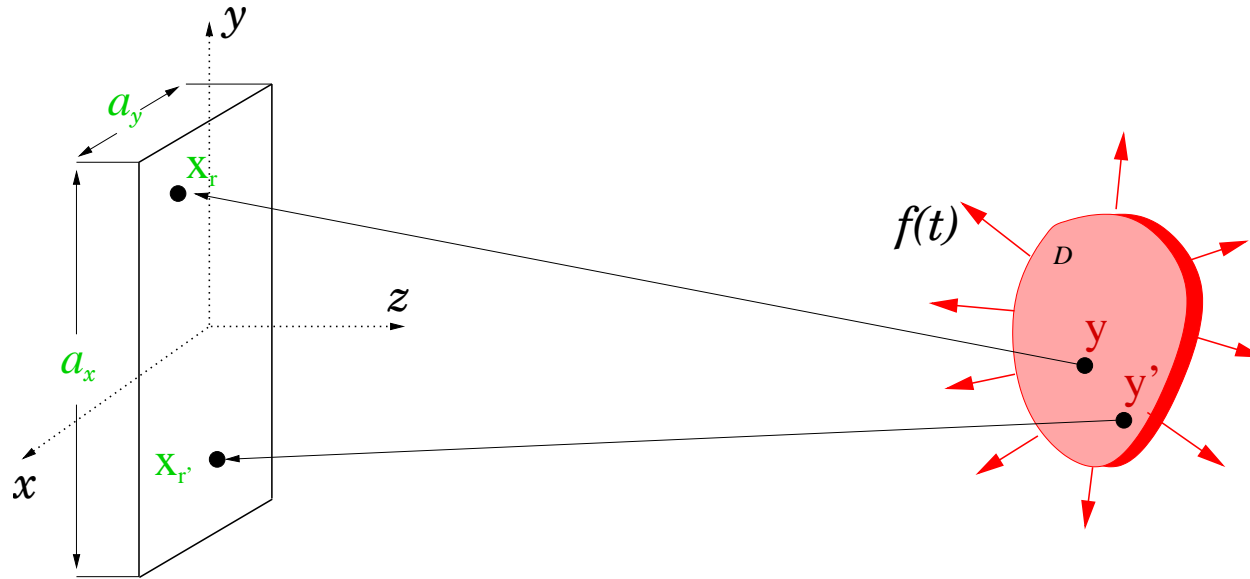
University of Chicago

In Collaboration with:

Liliana Borcea (Rice University)

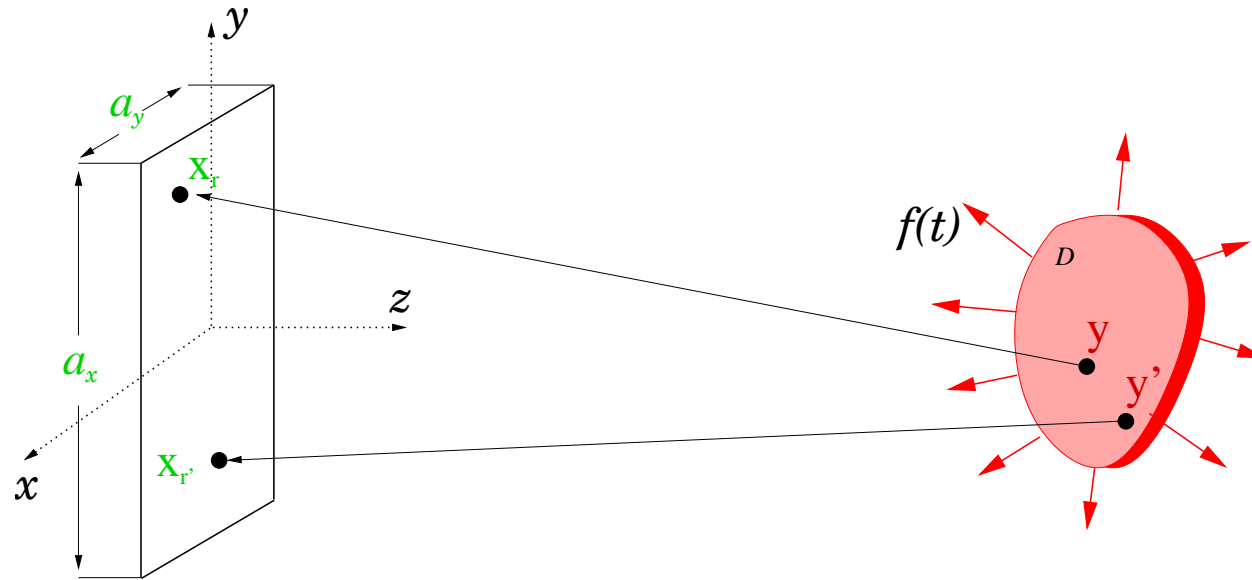
George Papanicolaou (Stanford University)

Passive Array Imaging in Clutter



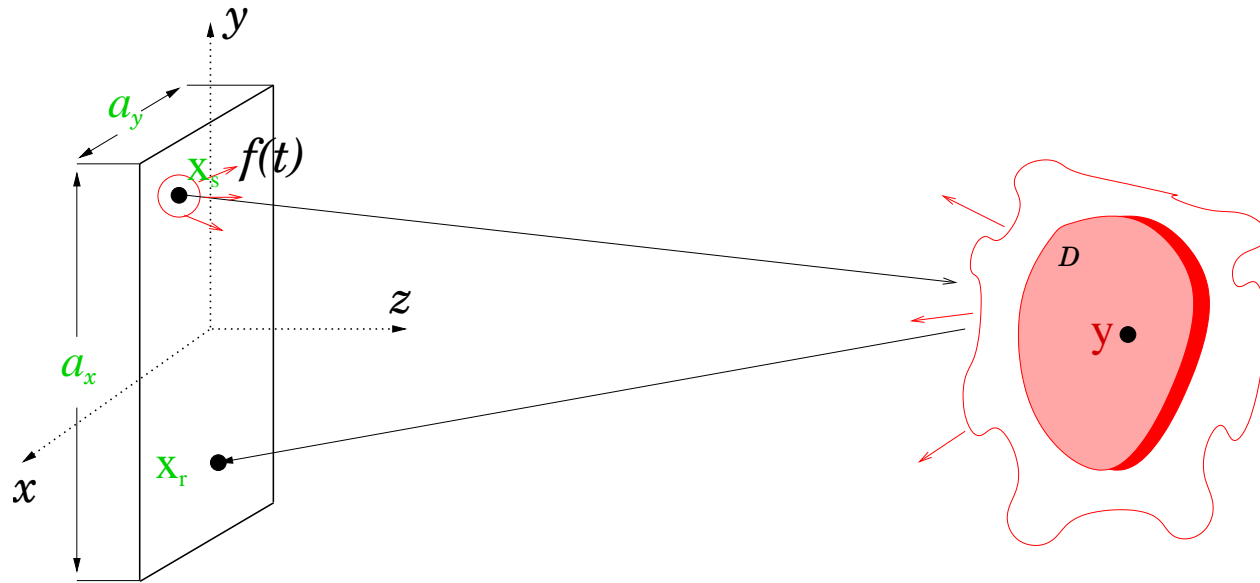
- **Array data:** $P(\mathbf{x}_r, t)$ for (\mathbf{x}_r, t) a set of receiver locations in \mathbb{R}^2 and time in \mathbb{R}_+ .
- **Object:** continuous distribution of sources in \mathcal{D} with intensity $\varrho(\mathbf{y})$.

Passive Array Imaging in Clutter



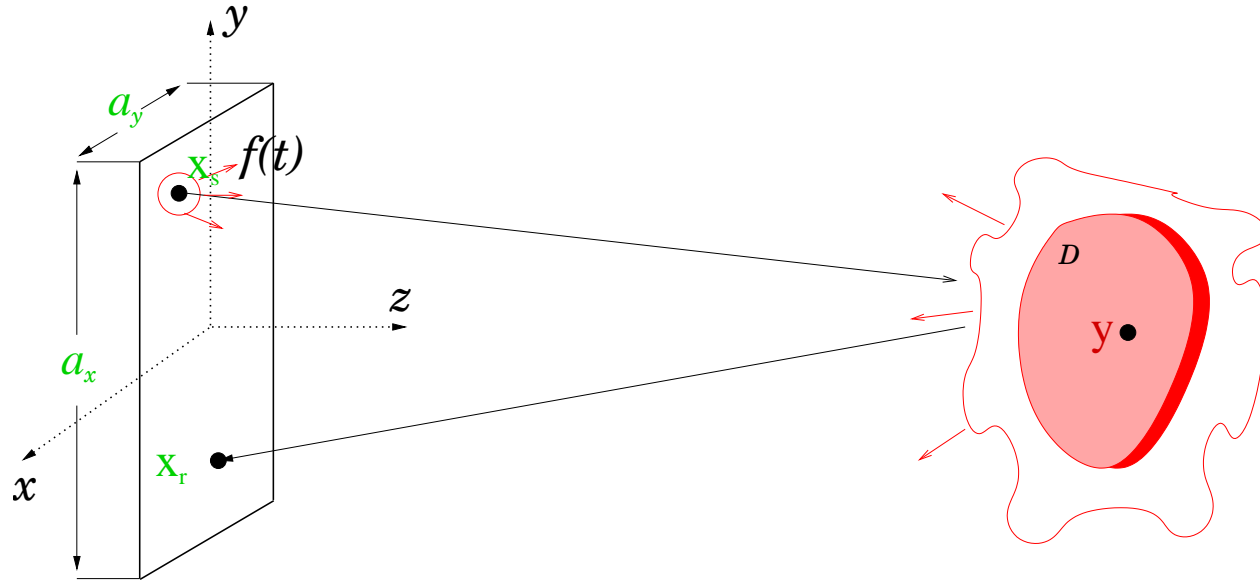
- **Objective:** recover \mathcal{D} from $P(\mathbf{x}_r, t)$ when the background medium is **cluttered**.

Active Array Imaging in Clutter



- **Array data:** $P(\mathbf{x}_s, \mathbf{x}_r, t)$ for $(\mathbf{x}_s, \mathbf{x}_r, t)$ a set of source and receiver locations in \mathbb{R}^2 and time in \mathbb{R}_+ .
- **Object:** scatterer with support in \mathcal{D} and reflectivity $\varrho(\mathbf{y})$.

Active Array Imaging in Clutter



- **Objective:** recover \mathcal{D} from $P(\mathbf{x}_s, \mathbf{x}_r, t)$ when the background medium is **cluttered**.
- **Application:** Imaging underground structures / Non-destructive testing (concrete imaging).

What is the clutter?

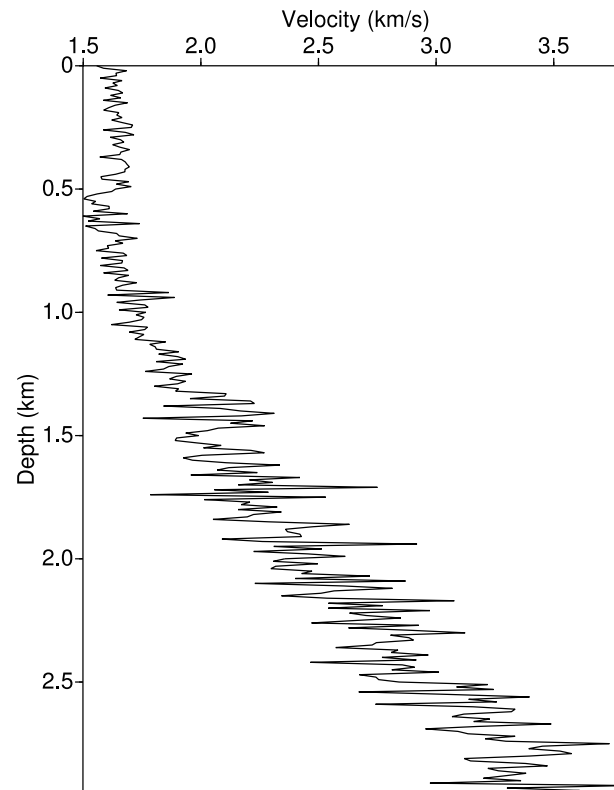
- background velocity $c(\mathbf{x})$ consists of a smooth part $c_o(\mathbf{x})$, that is **known** or can be estimated, and the inhomogeneities (**clutter**) that cannot be precisely estimated \Rightarrow model as random process.

$$c(\mathbf{x}) = c_o(\mathbf{x}) \left(1 + \sigma \mu \left(\frac{x_1}{l_1}, \frac{x_2}{l_2} \right) \right)$$

- with μ a random process
- l_1, l_2 the correlation lengths (scale of the inhomogeneities)

Velocity profile in the earth

- background velocity $c(\mathbf{x})$ consists of a smooth part $c_o(\mathbf{x})$ (assumed **known**), and of the fluctuations, which cannot be estimated.



- Velocity profile in a well log

Modeling the clutter

- We assume that the velocity is described by

$$c(\mathbf{x}) = c_0(\mathbf{x}) (1 + \sigma\mu(\mathbf{x}))$$

- with μ a real valued random process with $\langle \mu \rangle = 0$ and correlation function:

$$R(\mathbf{x}_1, \mathbf{x}_2) = \langle \mu(\mathbf{x}_1)\mu(\mathbf{x}_2) \rangle$$

- or by introducing $\bar{\mathbf{x}} = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}$, $\tilde{\mathbf{x}} = \mathbf{x}_2 - \mathbf{x}_1$

$$R(\bar{\mathbf{x}}, \tilde{\mathbf{x}}) = \langle \mu(\bar{\mathbf{x}} - \tilde{\mathbf{x}}/2)\mu(\bar{\mathbf{x}} + \tilde{\mathbf{x}}/2) \rangle$$

- We assume that μ is stationary so that the correlation function depends only on the distance

$$R(\bar{\mathbf{x}}, \tilde{\mathbf{x}}) = R(\tilde{\mathbf{x}})$$

Synthetic realization of random media

- on a rectangular grid we generate a filter $F(\mathbf{x})$
- we compute the Fourier transform $\hat{F}(\mathbf{k})$ of $F(\mathbf{x})$
- we generate a white noise distribution $\hat{W}(\mathbf{k})$
($\langle \hat{W} \rangle = 0$, $\text{std}=1$)
- we compute $\mu(\mathbf{x}) = \mathcal{F}^{-1}(\hat{W} \hat{F})$
- the correlation function of $\mu(\mathbf{x})$ is
($\langle \overline{\hat{W}(\mathbf{k}_1)} \hat{W}(\mathbf{k}_2) \rangle = \delta(\mathbf{k}_1 - \mathbf{k}_2)$)

$$R(\tilde{\mathbf{x}}) = (2\pi)^{-d} \int d\mathbf{k} e^{i\mathbf{k} \cdot \tilde{\mathbf{x}}} \overline{\hat{F}(\mathbf{k})} \hat{F}(\mathbf{k})$$

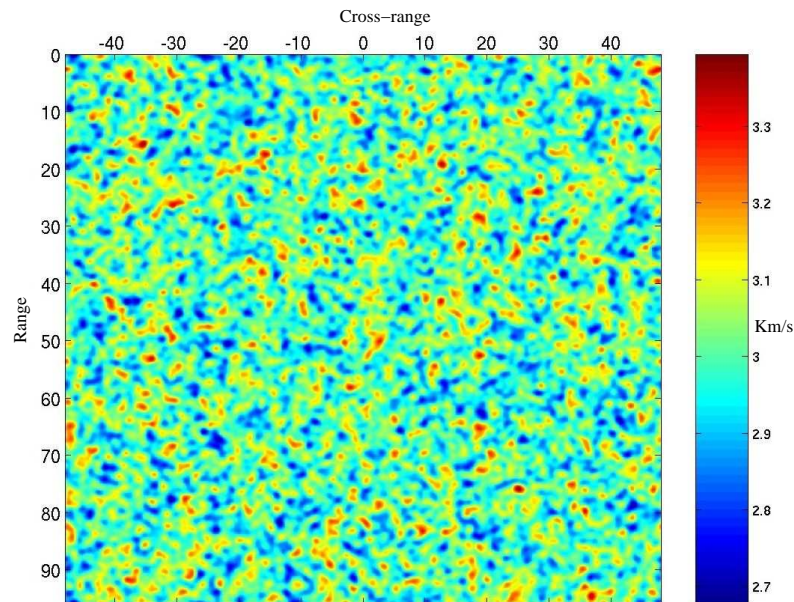
- we chose F to obtain the desired R

Synthetic realization of random media

- Examples of isotropic clutter correlation functions
- Gaussian $R(|\mathbf{x}_1 - \mathbf{x}_2|) = e^{-\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{2l^2}}$
- Power law $R(|\mathbf{x}_1 - \mathbf{x}_2|) = \left(1 + \frac{|\mathbf{x}_1 - \mathbf{x}_2|}{l}\right) e^{-\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{l}}$

Synthetic realization of random media

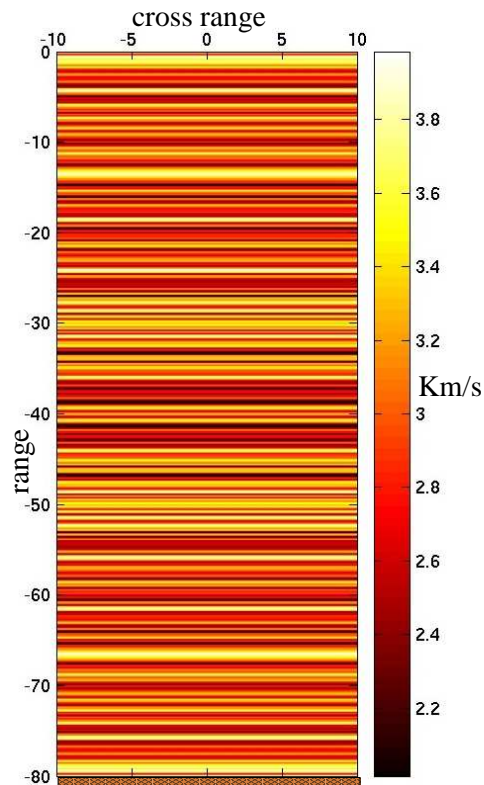
- 2d example with gaussian correlation fct



- here the correlation length l is the same in all directions of propagation ($l_{cr} = l_r = l$)

Synthetic realization of random media

- 1d example : Anisotropic (layered) clutter



- here the correlation length is infinite in the cross-range direction and finite in the range direction $l_r = l$

The forward model

- Data: $\hat{P}(\mathbf{x}_s, \mathbf{x}_r, \omega) = \hat{f}(\omega)\hat{G}(\mathbf{x}_s, \mathbf{x}_r, \omega)$
- $\hat{G}(\mathbf{x}, \mathbf{y}, \omega)$ satisfying the wave equation

$$\Delta\hat{G}(\mathbf{x}, \mathbf{y}, \omega) + k^2n^2(\mathbf{x})\hat{G}(\mathbf{x}, \mathbf{y}, \omega) = -\delta(\mathbf{x} - \mathbf{y}) \text{ in } \mathbb{R}^3$$

- $k = \omega/c_0$: wavenumber
- $n(\mathbf{x}) = c_0/c(\mathbf{x})$: index of refraction

$$n^2(\mathbf{x}) = n_{BG}^2(\mathbf{x}) + \varrho(\mathbf{x}) + \mu(\mathbf{x})$$

- μ = random part of the refractive index.

The inverse problem

- we can formulate the non-linear least squares problem:
Find $\varrho(\mathbf{x})$ by minimizing,

$$J(\varrho) = \int_0^T dt \sum_{\mathbf{x}_s, \mathbf{x}_r} |P(\mathbf{x}_s, \mathbf{x}_r, t) - Q(\mathbf{x}_s, \mathbf{x}_r, t; \varrho)|^2$$

- with $Q(\mathbf{x}_s, \mathbf{x}_r, t; \varrho)$ the data model.
- this is typically not solvable for large array data (as in seismic applications)

Linearized inversion

- Introduce \hat{G}_B solution of

$$\Delta \hat{G}_B(\mathbf{x}, \mathbf{y}, \omega) + k^2 (n_{BG}^2(\mathbf{x}) + \mu(\mathbf{x})) \hat{G}_B(\mathbf{x}, \mathbf{y}, \omega) = -\delta(\mathbf{x} - \mathbf{y}) \text{ in } \mathbb{R}^3$$

- the pressure field is given by,

$$\hat{P}(\mathbf{x}, \mathbf{y}, \omega) = \hat{f}(\omega) \hat{G}_B(\mathbf{x}, \mathbf{y}, \omega) + \hat{q}(\mathbf{x}, \mathbf{y}, \omega)$$

- with $\hat{q}(\mathbf{x}, \mathbf{y}, \omega)$ solution of,

$$\begin{aligned} (\Delta + k^2 (n_{BG}^2(\mathbf{x}) + \mu(\mathbf{x}))) \hat{q}(\mathbf{x}, \mathbf{y}, \omega) = \\ -k^2 \varrho(\mathbf{x}) (\hat{f}(\omega) \hat{G}_B(\mathbf{x}, \mathbf{y}, \omega) + \hat{q}(\mathbf{x}, \mathbf{y}, \omega)) \end{aligned}$$

Linearized inversion

- So that,

$$\hat{q}(\mathbf{x}, \mathbf{y}, \omega) = -k^2 \hat{f}(\omega) \int \varrho(\mathbf{z}) \hat{G}_B(\mathbf{x}, \mathbf{z}, \omega) \hat{G}_B(\mathbf{z}, \mathbf{y}, \omega) d\mathbf{z} \\ - k^2 \int \varrho(\mathbf{z}) \hat{q}(\mathbf{x}, \mathbf{z}, \omega) \hat{G}_B(\mathbf{z}, \mathbf{y}, \omega) d\mathbf{z}$$

- Linearization consists in (Born approximation)

$$\hat{q}(\mathbf{x}, \mathbf{y}, \omega) = -k^2 \hat{f}(\omega) \int \varrho(\mathbf{z}) \hat{G}_B(\mathbf{x}, \mathbf{z}, \omega) \hat{G}_B(\mathbf{z}, \mathbf{y}, \omega) d\mathbf{z}$$

Kirchhoff migration

- Let's assume that we know $n_{BG}^2(\mathbf{x})$, $\mu(\mathbf{x}) = 0$.
- The solution of the linearized least squares problem:

$$J(\boldsymbol{\rho}) = \int d\omega \sum_{\mathbf{x}_s, \mathbf{x}_r} |\hat{P}(\mathbf{x}_s, \mathbf{x}_r, \omega) - \hat{Q}_L(\mathbf{x}_s, \mathbf{x}_r, \omega; \boldsymbol{\rho})|^2$$

- with $Q_L(\mathbf{x}_s, \mathbf{x}_r, \omega; \boldsymbol{\rho}) = \mathcal{A}\boldsymbol{\rho}$

$$\hat{Q}_L(\mathbf{x}_s, \mathbf{x}_r, \omega; \boldsymbol{\rho}) = -k^2 \hat{f}(\omega) \int \boldsymbol{\rho}(\mathbf{z}) \hat{G}_B(\mathbf{x}, \mathbf{z}, \omega) G_B(\mathbf{z}, \mathbf{y}, \omega) d\mathbf{z}$$

- is given by $\boldsymbol{\rho} = \mathcal{A}^* P(\mathbf{x}_s, \mathbf{x}_r, t)$ because $\mathcal{A}^* \mathcal{A}$ acts as an identity operator on the singularities of $\boldsymbol{\rho}$.

Kirchhoff migration

- assuming $n_{BG}^2(\mathbf{x})$ is smooth and using HF asymptotics for the Green's function (neglecting amplitudes) we get that $\mathcal{I}^{KM}(\mathbf{y}^s)$ gives a good estimate of $q(\mathbf{y}^s)$ (NOTE: we only recover the support - singularities of the function)

$$\mathcal{I}^{KM}(\mathbf{y}^s) = \sum_{\mathbf{x}_s, \mathbf{x}_r} \int d\omega \hat{P}(\mathbf{x}_s, \mathbf{x}_r, \omega) e^{-i\omega(\tau(\mathbf{x}_s, \mathbf{y}^s) + \tau(\mathbf{y}^s, \mathbf{x}_r))}$$

- $\tau(\mathbf{x}, \mathbf{y})$ is the travel time $\tau(\mathbf{x}, \mathbf{y}) = \min \int \frac{1}{c(X(s))} ds$ where the minimum is over all paths X that start at \mathbf{x} and end at \mathbf{y} .

Kirchhoff migration

● references:

- N. Bleistein, J.K. Cohen, and J.W. Stockwell Jr., *Mathematics of multidimensional seismic imaging, migration, and inversion*. Springer, New York, 2001.
- G. Beylkin and R. Burridge. *Linearized inverse scattering problems in acoustics and elasticity*. Wave Motion, Vol. 12, No. 1, pp. 15-52, 1990.
- Lewis and Symes, *Inverse Problems*, Vol. 7, pp. 597-632, 1991.
- W. Symes. *Lecture notes in seismic imaging*. Mathematical Geophysics Summer School, Stanford, available at www.trip.caam.rice.edu, 1998.
- C. Stolk and M. V. de Hoop. *Microlocal analysis of seismic inverse scattering in anisotropic elastic media*. Comm. Pure Appl. Math., Vol. 55, No. 3, pp. 261-301, 2002.
- C. Stolk and W. Symes. *Smooth objective functionals for seismic velocity inversion*. Inverse Problems, Vol. 19, pp. 73-89, 2003.

Kirchhoff migration resolution

- when $a, B \rightarrow \infty \Rightarrow$

$$\mathcal{I}^{\text{KM}}(\mathbf{y}^S) \approx \int_{\mathcal{D}} \delta(\mathbf{y} - \mathbf{y}^S) \varrho(\mathbf{y}) d\mathbf{y}$$

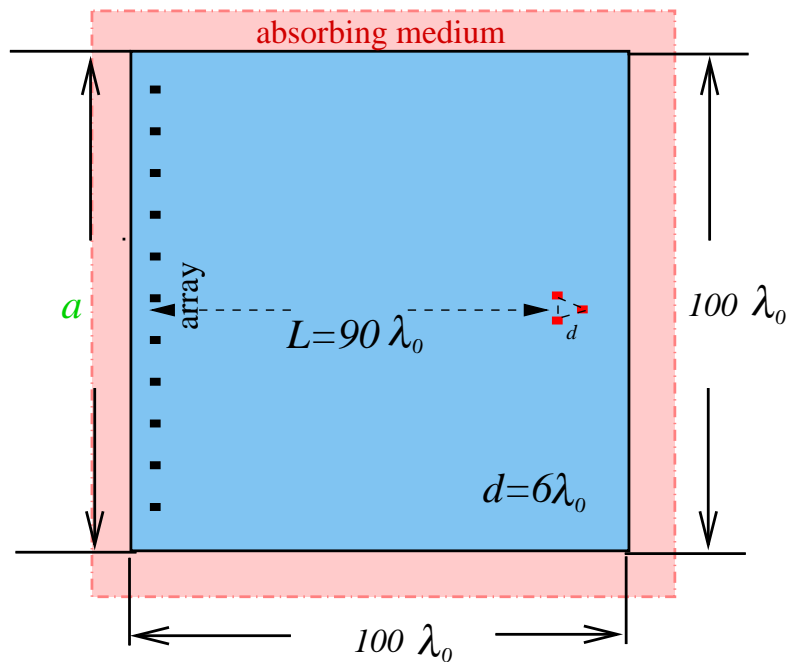
Bleinstein, Cohen, Stockwell (2001)

- for finite a and $B \Rightarrow$

- range resolution (direction of propagation): $\sigma_r = \frac{c_0}{B}$
- cross-range resolution: $\sigma_{\text{cr}} = \frac{\lambda_0 L}{a}$

The numerical setup

Length scaled by λ_0



$$\rho(\mathbf{x}) \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0$$

$$\kappa(\mathbf{x}) \frac{\partial p}{\partial t} + \text{div} \mathbf{v} = f(t) \delta(\mathbf{x} - \mathbf{x}_s)$$

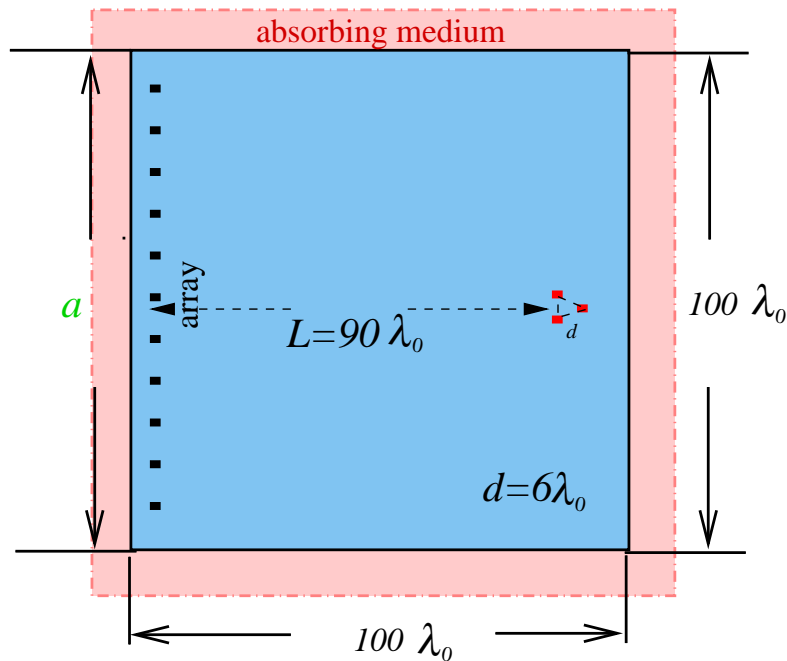
$$\rho(\mathbf{x}) = 1$$

$$\kappa(\mathbf{x}) = \frac{1}{\rho c^2(\mathbf{x})}$$

- the background velocity is $c_0 = 3\text{km}/\text{sec}$
- $f(t)$: is the derivative of a gaussian with central frequency $f_0 = 100\text{kHz}$ and bandwidth $60 - 130\text{kHz}$ measured at 6dB.

The numerical setup

Length scaled by λ_0

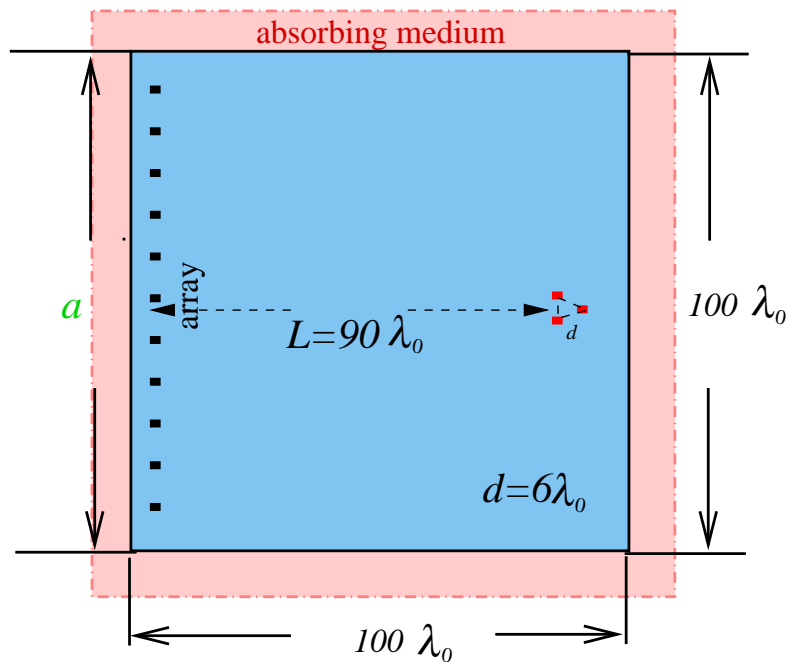


$$\begin{aligned}\rho(\mathbf{x}) \frac{\partial \mathbf{v}}{\partial t} + \nabla p &= 0 \\ \kappa(\mathbf{x}) \frac{\partial p}{\partial t} + \text{div} \mathbf{v} &= f(t) \delta(\mathbf{x} - \mathbf{x}_s) \\ \rho(\mathbf{x}) &= 1 \\ \kappa(\mathbf{x}) &= \frac{1}{\rho c^2(\mathbf{x})}\end{aligned}$$

- the central wavelength is $\lambda_0 = 3\text{cm}$.
- array: 185 elements $\lambda_0/2$ apart, $a = 92\lambda_0$
- the range is $90\lambda_0$

The numerical setup

Length scaled by λ_0



$$\rho(\mathbf{x}) \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0$$

$$\kappa(\mathbf{x}) \frac{\partial p}{\partial t} + \operatorname{div} \mathbf{v} = f(t) \delta(\mathbf{x} - \mathbf{x}_s)$$

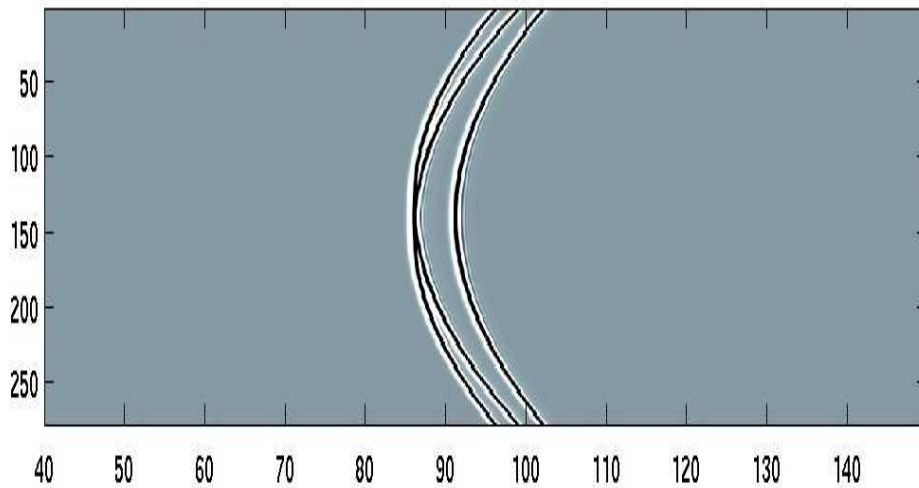
$$\rho(\mathbf{x}) = 1$$

$$\kappa(\mathbf{x}) = \frac{1}{\rho c^2(\mathbf{x})}$$

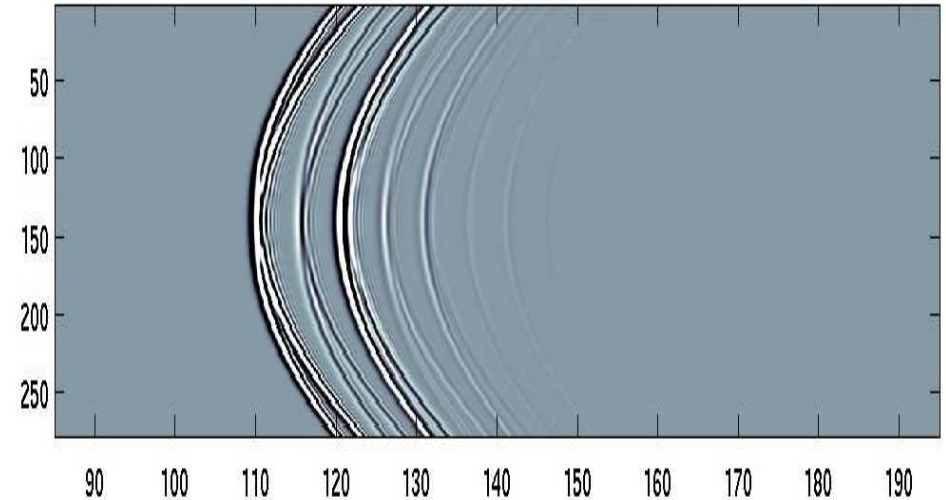
- the distance between the objects (sources or targets) is $d = 6 \lambda_0$ (or $3 \lambda_0$)
- the targets are disks with diameter λ_0 .

Data on the array: traces

Passive array



Active array



- the cross-range is measured in cm.
- the time is measured in msec.

Kirchhoff migration

- Passive array: imaging functional for KM at \mathbf{y}^S

$$\mathcal{I}^{\text{KM}}(\mathbf{y}^S) = \sum_r P(\mathbf{x}_r, \tau(\mathbf{x}_r, \mathbf{y}^S)) = \sum_r \int_B \frac{d\omega}{2\pi} \hat{P}(\mathbf{x}_r, \omega) e^{-i\omega\tau(\mathbf{x}_r, \mathbf{y}^S)}$$

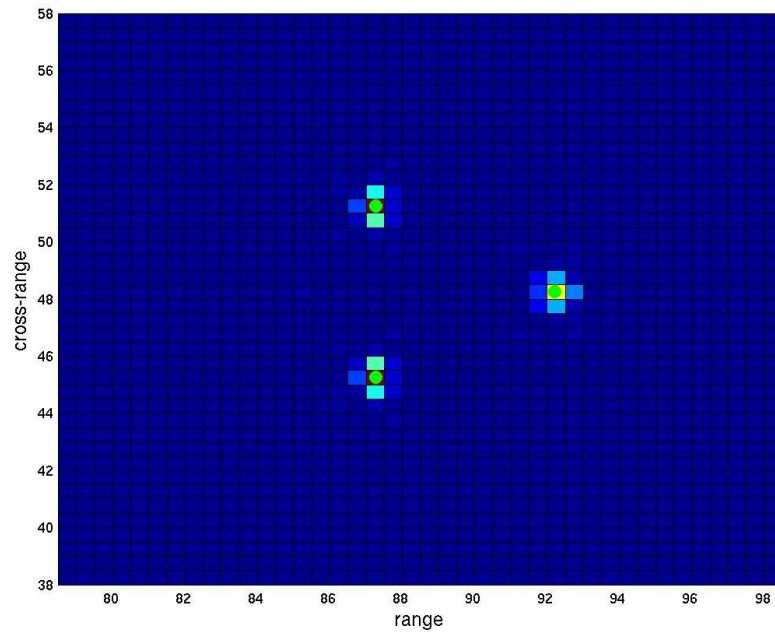
- with $\tau(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|/c_0$ travel time in the known smooth background (here homogeneous)
- Active array: imaging functional for KM at \mathbf{y}^S

$$\begin{aligned} \mathcal{I}^{\text{KM}}(\mathbf{y}^S) &= \sum_{r=1}^{N_r} P(\mathbf{x}_S, \mathbf{x}_r, \tau(\mathbf{x}_S, \mathbf{y}^S) + \tau(\mathbf{x}_r, \mathbf{y}^S)) \\ &= \sum_{r=1}^{N_r} \int \frac{d\omega}{2\pi} \hat{P}(\mathbf{x}_S, \mathbf{x}_r, \omega) \overline{G_0(\mathbf{x}_S, \mathbf{y}^S, \omega)} G_0(\mathbf{x}_r, \mathbf{y}^S, \omega) \end{aligned}$$

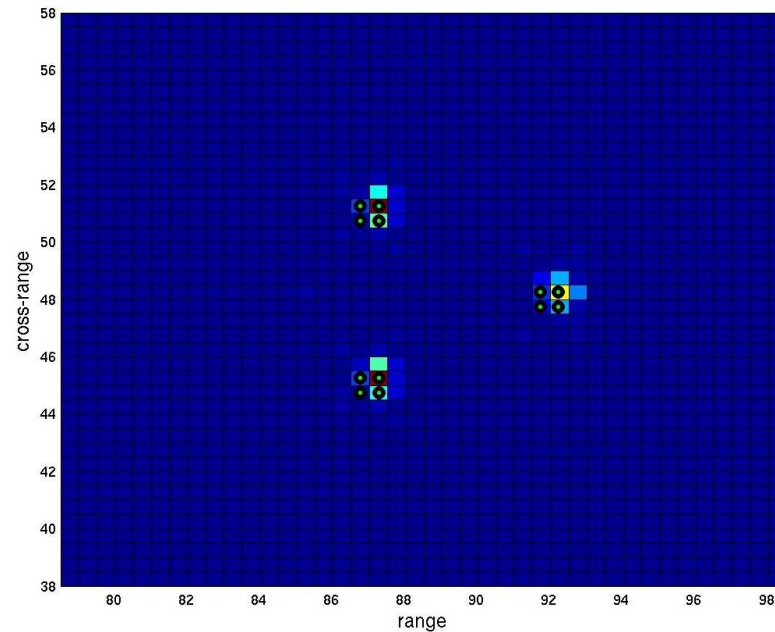
- with $G_0(\mathbf{x}_S, \mathbf{y}^S, \omega) = e^{i\omega\tau(\mathbf{x}_S, \mathbf{y}^S)}$

Kirchhoff migration results

Passive array



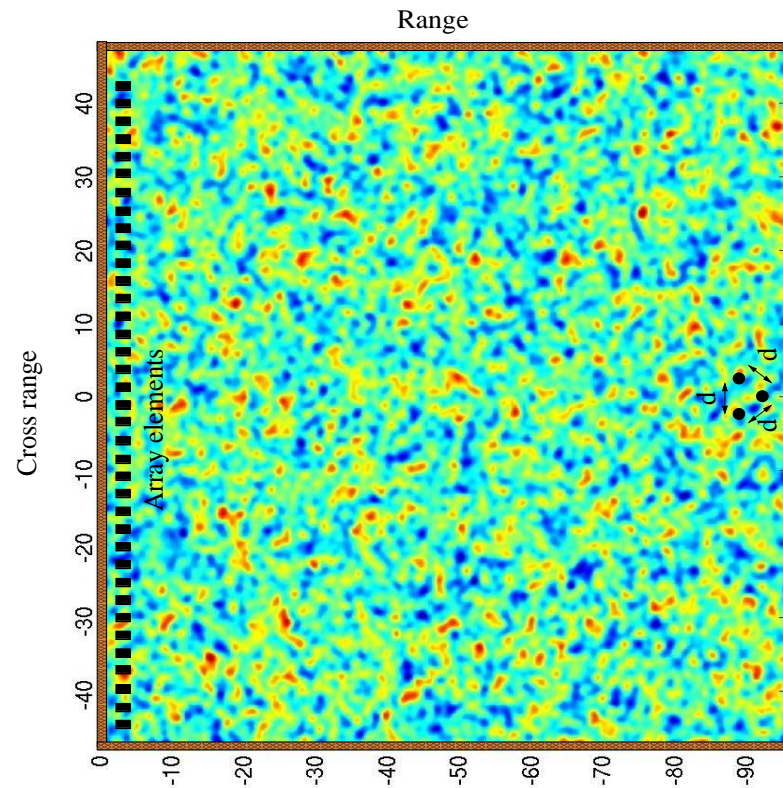
Active array



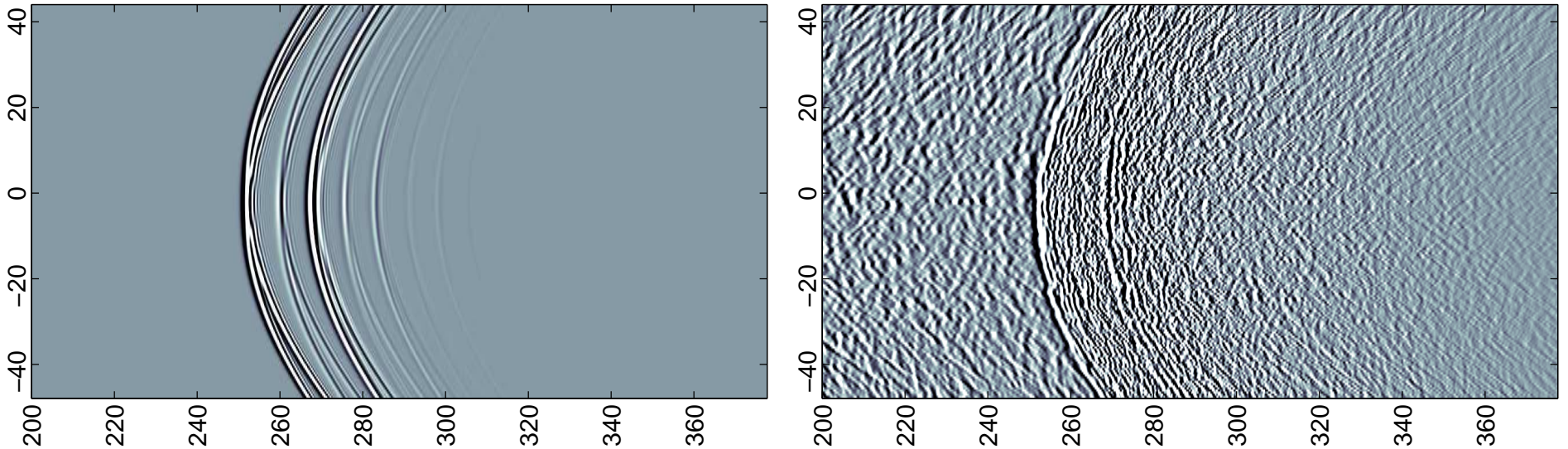
- length is scaled by λ_0
- the search domain is a square $20\lambda_0 \times 20\lambda_0$ centered at the objects
- the pixel size is $\lambda_0/2$.

What happens in clutter?

- Length scaled by λ_0



What happens in clutter?

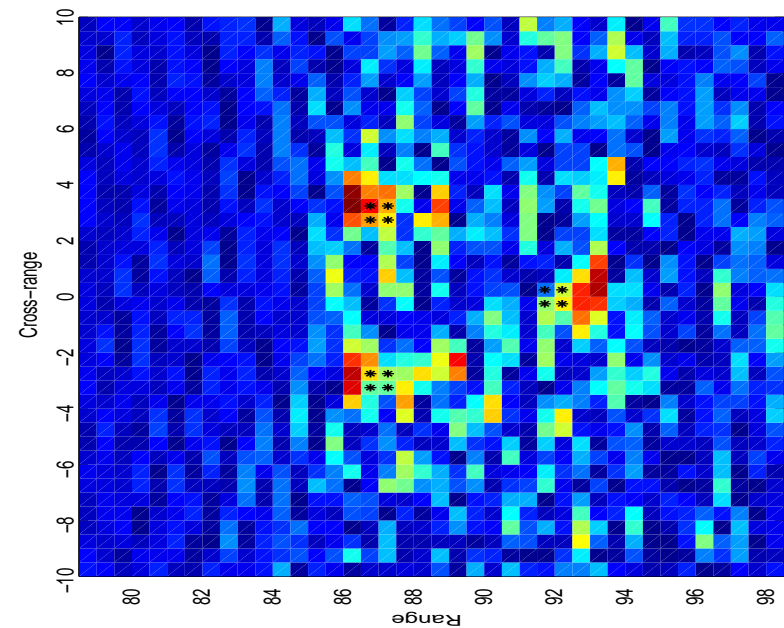
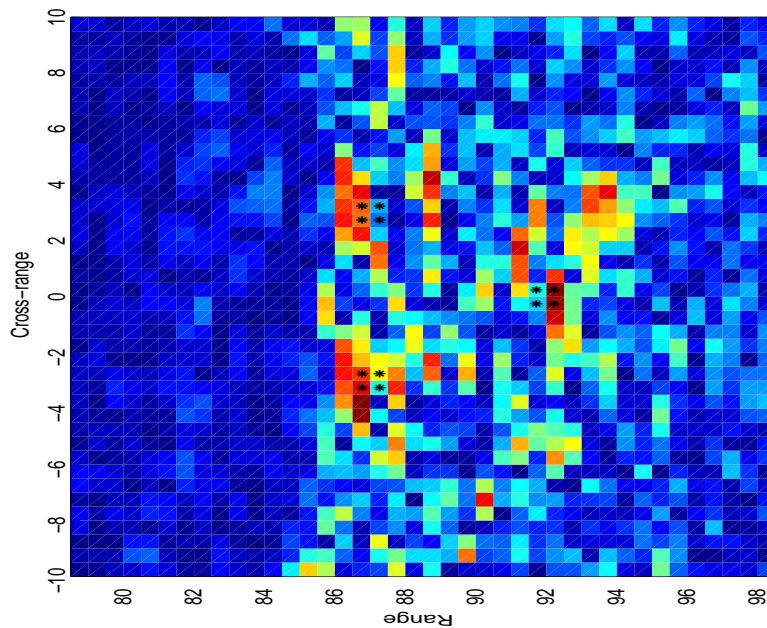


- Length scaled by λ_0 and time by pulsewidth
- the clutter impedes the imaging process as the significant multipathing of the waves by the inhomogeneities results to noisy data traces (the noise is not simply additive)

Migration in clutter

- Classic migration is statistically unstable

$$\mathcal{I}^{\text{KM}}(\mathbf{y}^s) = \sum_{r=1}^{N_r} P(\mathbf{x}_s, \mathbf{x}_r, \tau(\mathbf{x}_s, \mathbf{y}^s) + \tau(\mathbf{x}_r, \mathbf{y}^s))$$



Migration in clutter

- Classic migration is statistically unstable

$$\mathcal{I}^{\text{KM}}(\mathbf{y}^s) = \sum_{r=1}^{N_r} P(\mathbf{x}_s, \mathbf{x}_r, \tau(\mathbf{x}_s, \mathbf{y}^s) + \tau(\mathbf{x}_r, \mathbf{y}^s))$$

- To make migration work we should remove the delay spread:
 - ✗ trace denoising ? (noise is not additive)
 - ✓ we use time-reversal based techniques

Migration in frequency domain

- the migration functional

$$\mathcal{I}^{\text{KM}}(\mathbf{y}^s) = \sum_{r=1}^{N_r} P(\mathbf{x}_s, \mathbf{x}_r, \tau(\mathbf{x}_s, \mathbf{y}^s) + \tau(\mathbf{x}_r, \mathbf{y}^s))$$

- can be written as

$$\mathcal{I}^{\text{KM}}(\mathbf{y}^s) = \sum_{r=1}^{N_r} \int d\omega \hat{P}(\mathbf{x}_s, \mathbf{x}_r, \omega) \overline{G_0(\mathbf{x}_s, \mathbf{y}^s, \omega) G_0(\mathbf{x}_r, \mathbf{y}^s, \omega)}$$

- with $G_0(\mathbf{x}_s, \mathbf{y}^s, \omega) = e^{i\omega\tau(\mathbf{x}_s, \mathbf{y}^s)}$

Coherent interferometry (CINT)

- an ideal way to image would be to backpropagate with the exact $G(\mathbf{x}_s, \mathbf{y}^s, \omega)$. This is called time reversal and has two fundamental properties in clutter:
 - ✓ statistical stability
 - ✓ super-resolution
- But we do not know the clutter ! ($G(\mathbf{x}_s, \mathbf{y}^s, \omega)$ is unknow)

Coherent interferometry (CINT)

- we cross-correlate the traces locally in space and time:
 - cross-correlation in space is limited by the decoherence length X_d
 - cross-correlation in time is limited by the delay spread T_d
- we call these local cross-correlations coherent interferograms
- CINT consists in migrating the coherent interferograms to the search point \mathbf{y}^s using $G_0(\mathbf{x}_s, \mathbf{y}^s, \omega)$

CINT imaging functional

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^s; \Omega_d, \kappa_d) \sim \int_{\bar{\omega} \in B} d\bar{\omega} \int_{\bar{\mathbf{x}} \in a} d\bar{\mathbf{x}} \int d\tilde{\omega} \hat{\Psi}(\tilde{\omega}; \Omega_d) \int d\tilde{\mathbf{x}} \hat{\Phi}\left(\frac{\bar{\omega}}{c_0} \tilde{\mathbf{x}}; \kappa_d^{-1}\right) \\ \hat{P}\left(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, \bar{\omega} + \frac{\tilde{\omega}}{2}\right) \exp\left\{-i\left(\bar{\omega} + \frac{\tilde{\omega}}{2}\right) \left[\tau\left(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s\right) + \tau(\mathbf{x}_s, \mathbf{y}^s)\right]\right\} \\ \overline{\hat{P}\left(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, \bar{\omega} - \frac{\tilde{\omega}}{2}\right) \exp\left\{+i\left(\bar{\omega} - \frac{\tilde{\omega}}{2}\right) \left[\tau\left(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s\right) + \tau(\mathbf{x}_s, \mathbf{y}^s)\right]\right\}}$$

using the midpoint and offset variables

$$\bar{\mathbf{x}} = \frac{\mathbf{x}_r + \mathbf{x}_r'}{2}, \tilde{\mathbf{x}} = \mathbf{x}_r - \mathbf{x}_r'; \quad \bar{\omega} = \frac{\omega + \omega'}{2}, \tilde{\omega} = \omega - \omega'$$

CINT imaging functional

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^s; \Omega_d, \kappa_d) \sim \int_{\bar{\omega} \in B} d\bar{\omega} \int_{\bar{\mathbf{x}} \in a} d\bar{\mathbf{x}} \int d\tilde{\omega} \hat{\Psi}(\tilde{\omega}; \Omega_d) \int d\tilde{\mathbf{x}} \hat{\Phi}\left(\frac{\bar{\omega}}{c_0} \tilde{\mathbf{x}}; \kappa_d^{-1}\right) \\ \hat{P}\left(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, \bar{\omega} + \frac{\tilde{\omega}}{2}\right) \exp\left\{-i\left(\bar{\omega} + \frac{\tilde{\omega}}{2}\right) \left[\tau\left(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s\right) + \tau(\mathbf{x}_s, \mathbf{y}^s)\right]\right\} \\ \overline{\hat{P}\left(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, \bar{\omega} - \frac{\tilde{\omega}}{2}\right) \exp\left\{+i\left(\bar{\omega} - \frac{\tilde{\omega}}{2}\right) \left[\tau\left(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s\right) + \tau(\mathbf{x}_s, \mathbf{y}^s)\right]\right\}}$$

- $\tilde{\omega}$ is restricted by window $\hat{\Psi}$ to $|\tilde{\omega}| \leq \Omega_d$, with Ω_d the decoherence frequency ($\sim 1/T_d$)
- $\tilde{\mathbf{x}}$ is restricted by window $\hat{\Phi}$ to $|\tilde{\mathbf{x}}| \leq X_d(\bar{\omega})$, with $X_d(\bar{\omega})$ the decoherence length (the TR spot size at frequency $\bar{\omega}$). The support of $\hat{\Phi}$ is $\kappa_d^{-1} = \bar{\omega} X_d(\bar{\omega}) / c_0$

CINT and statistical smoothing

- for small $|\tilde{\mathbf{x}}|$ we can linearize the phase

$$\exp \left\{ -i\bar{\omega} \left[\tau(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s) - \tau(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s) \right] \right\} \approx \exp \left\{ -i\bar{\omega}\tilde{\mathbf{x}} \cdot \nabla_{\bar{\mathbf{x}}}\tau(\bar{\mathbf{x}}, \mathbf{y}^s) \right\}$$

$$\exp \left\{ -i\tilde{\omega} \left[\tau(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s) + \tau(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s) \right] \right\} \approx \exp \left\{ -i2\tilde{\omega}\tau(\bar{\mathbf{x}}, \mathbf{y}^s) \right\}$$

CINT and statistical smoothing

- the imaging functional becomes

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^S; \Omega_d, \kappa_d) = \int dt \int d\mathbf{k} \Phi(c_0 \nabla_{\bar{\mathbf{x}}} \tau(\bar{\mathbf{x}}, \mathbf{y}^S) - \mathbf{k}; \kappa_d) \\ \Psi(\tau(\bar{\mathbf{x}}, \mathbf{y}^S) + \tau(\mathbf{x}_s, \mathbf{y}^S) - t; T_d) \int d\bar{\omega} W\left(\bar{\mathbf{x}}, \frac{\bar{\omega}}{c_0} \mathbf{k}, t\right),$$

- with $W(\cdot)$ the Wigner distribution of the data

$$W\left(\bar{\mathbf{x}}, \frac{\bar{\omega}}{c_0} \mathbf{k}, t\right) = \frac{\int d\tilde{\omega} \int d\tilde{\mathbf{x}} e^{-i\tilde{\omega}t - i\frac{\bar{\omega}}{c_0}\tilde{\mathbf{x}} \cdot \mathbf{k}} \hat{P}\left(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, \bar{\omega} + \frac{\tilde{\omega}}{2}\right)}{\hat{P}\left(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, \bar{\omega} - \frac{\tilde{\omega}}{2}\right)}$$

- $W(\cdot)$ is highly fluctuating but decorrelates rapidly in $\bar{\omega}$ and \mathbf{k}
→ in CINT we have stability by smoothing

CINT as smooth migration

- CINT can be also written as

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^S; \Omega_d, \kappa_d) = \int d\bar{\mathbf{x}} \int d\tilde{\mathbf{x}} \left[P \left(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, t + \frac{\mathbf{k} \cdot \tilde{\mathbf{x}}}{2c_0} \right) P \left(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, t - \frac{\mathbf{k} \cdot \tilde{\mathbf{x}}}{2c_0} \right) \right]$$

$$\star_{\mathbf{k}} \Phi(\mathbf{k}; \kappa_d) \Big|_{\mathbf{k}=c_0 \nabla_{\bar{\mathbf{x}}}\tau(\bar{\mathbf{x}}, \mathbf{y}^s)} \star_t \Psi(t; T_d) \Big|_{t=\tau(\bar{\mathbf{x}}, \mathbf{y}^s) + \tau(\mathbf{x}_s, \mathbf{y}^s)}$$

- when Φ, Ψ are δ functions (no smoothing) we obtain

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^S; \Omega_d, \kappa_d) = [\mathcal{I}^{\text{KM}}(\mathbf{y}^s)]^2$$

- CINT is a statistically stable smoothed migration method !

CINT as smooth migration

- CINT can be also written as

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^S; \Omega_d, \kappa_d) = \int d\bar{\mathbf{x}} \int d\tilde{\mathbf{x}} \left[P \left(\bar{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, t + \frac{\mathbf{k} \cdot \tilde{\mathbf{x}}}{2c_0} \right) P \left(\bar{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, \mathbf{x}_s, t - \frac{\mathbf{k} \cdot \tilde{\mathbf{x}}}{2c_0} \right) \right]$$

$$\star_{\mathbf{k}} \Phi(\mathbf{k}; \kappa_d) \Big|_{\mathbf{k}=c_0 \nabla_{\bar{\mathbf{x}}} \tau(\bar{\mathbf{x}}, \mathbf{y}^s)} \star_t \Psi(t; T_d) \Big|_{t=\tau(\bar{\mathbf{x}}, \mathbf{y}^s) + \tau(\mathbf{x}_s, \mathbf{y}^s)}$$

- Smoothing over arrival time by convolution with $\Psi(t; T_d)$ of support $T_d \approx 1/\Omega_d$ affects range resolution c_0/Ω_d .
- Smoothing in direction of arrival by convol. with $\Phi(\mathbf{k}; \kappa_d)$ with supp. in ball of radius $\kappa_d \rightsquigarrow$ cross range resolution $L\kappa_d \approx \lambda_0 L/X_d(\omega_0)$.

Resolution summary

- migration resolution in homogeneous media
 - in range : $O\left(\frac{c_0}{B}\right)$
 - in cross-range : $O\left(\lambda\frac{L}{a}\right) = O\left(\frac{c_0L}{\omega_0 a}\right)$
- CINT resolution in clutter ($\Omega_d < B$ & $X_d < a$)
 - in range : $O\left(\frac{c_0}{\Omega_d}\right)$
 - in cross-range : $O(L\kappa_d) = O\left(\frac{c_0L}{\omega_0 X_d(\omega_0)}\right)$
- for $\Omega_d \ll B$ & $X_d \ll a$
 - ✓ incoherent imaging should be used (diffusion)
$$D = \frac{c_0 l^*}{3}$$
 - CINT works for $L < l^*$ (in numerics $l^* = 75\lambda_0$)

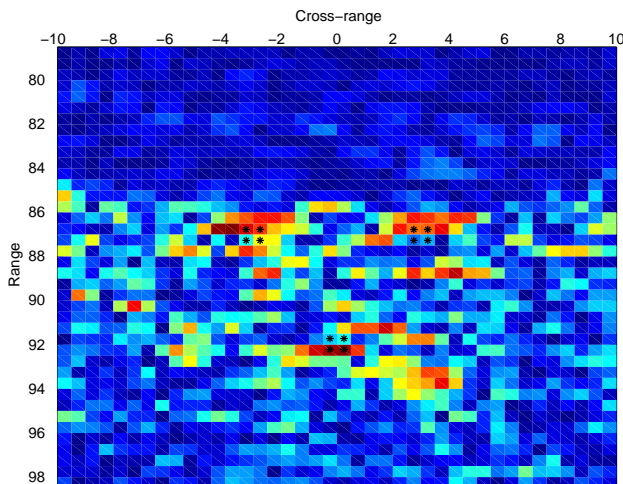
Adaptive CINT

- How can we find Ω_d and κ_d ?
- We may derive (theoretical) formulae for Ω_d and κ_d .
But this will be model dependent.
- We can estimate the decoherence parameters using statistical data processing techniques, but this can be tricky.
- We found that a more efficient approach is to do an adaptive estimation of the smoothing parameters, during the image formation process.

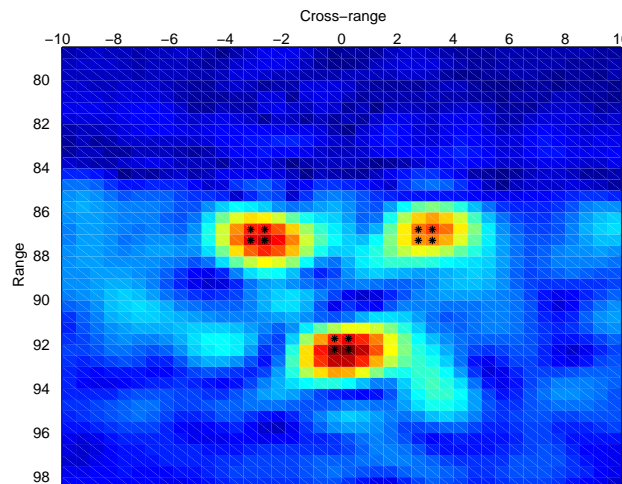
Adaptive CINT

- View the imaging function as $\mathcal{I}^{\text{CINT}}(\mathbf{y}^s; \Omega_d, \kappa_d)$ and seek parameters Ω_d and κ_d by achieving an optimal balance between statistical smoothing and resolution.
- Penalize the speckles (left image) by using a norm of the gradient. To obtain a tight image, we should also penalize the blur (see right image) by using a sparsity measure. The “optimal” result is given in the middle.

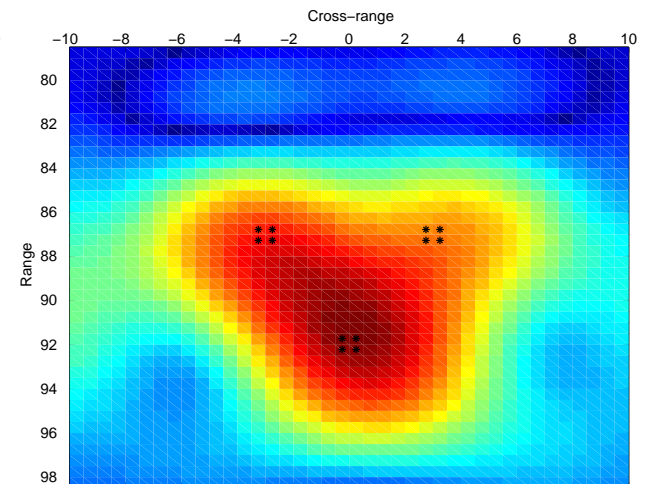
$$X_d = a, \Omega_d = B$$



$$X_d = X_d^*, \Omega_d = \Omega_d^*$$



$$X_d < X_d^*, \Omega_d < \Omega_d^*$$



Adaptive CINT

- Ω_d and κ_d are determined by minimizing

$$\mathcal{O}(\mathbf{y}^s; \Omega_d, \kappa_d) =$$

$$\|\mathcal{J}_{\mathcal{N}}(\mathbf{y}^s; \Omega_d, \kappa_d)\|_{L^1(\mathcal{D})} + \alpha \|\nabla_{\mathbf{y}^s} \mathcal{J}_{\mathcal{N}}(\mathbf{y}^s; \Omega_d, \kappa_d)\|_{L^1(\mathcal{D})},$$

- with $\mathcal{J}_{\mathcal{N}}(\mathbf{y}^s) = \sqrt{|\mathcal{J}(\mathbf{y}^s)|} / \sup_{\mathbf{y}^s \in \mathcal{D}_s} \sqrt{|\mathcal{J}(\mathbf{y}^s)|}$

- for point targets we use $\alpha = 1$

Adaptive CINT

- Ω_d and κ_d are determined by minimizing

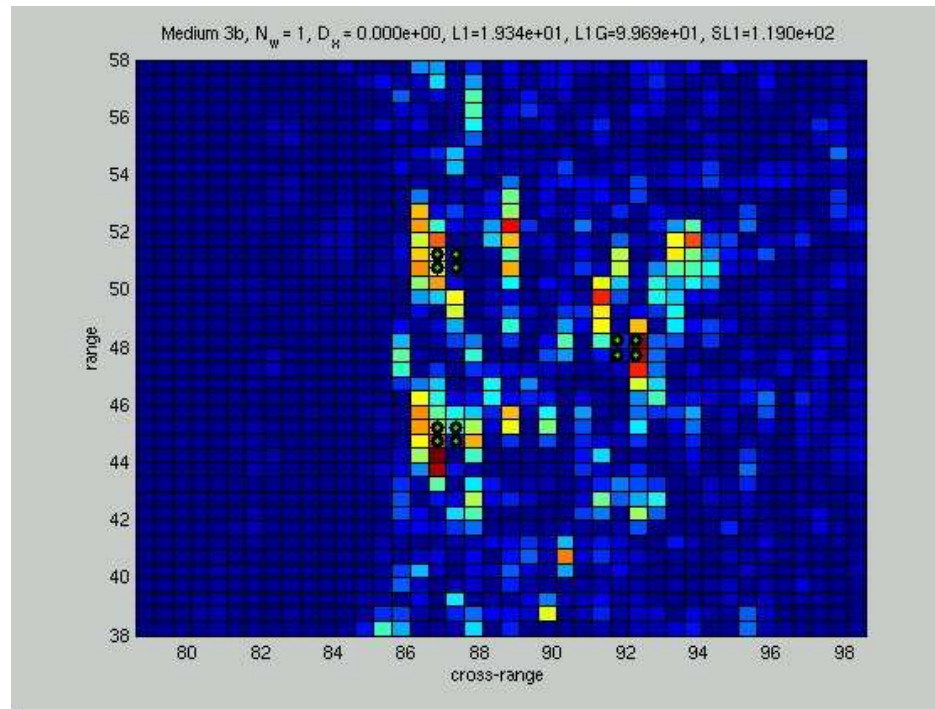
$$\mathcal{O}(\mathbf{y}^s; \Omega_d, \kappa_d) = \|\mathcal{J}_{\mathcal{N}}(\mathbf{y}^s; \Omega_d, \kappa_d)\|_{L^1(\mathcal{D})} + \alpha \|\nabla_{\mathbf{y}^s} \mathcal{J}_{\mathcal{N}}(\mathbf{y}^s; \Omega_d, \kappa_d)\|_{L^1(\mathcal{D})},$$

- This is very different from usual denoising,

$$\|\mathcal{N}(\mathbf{y}^s) - \mathcal{I}(\mathbf{y}^s)\|_{\text{prox}} + \alpha \|\mathcal{I}(\mathbf{y}^s)\|_{\text{reg}}$$

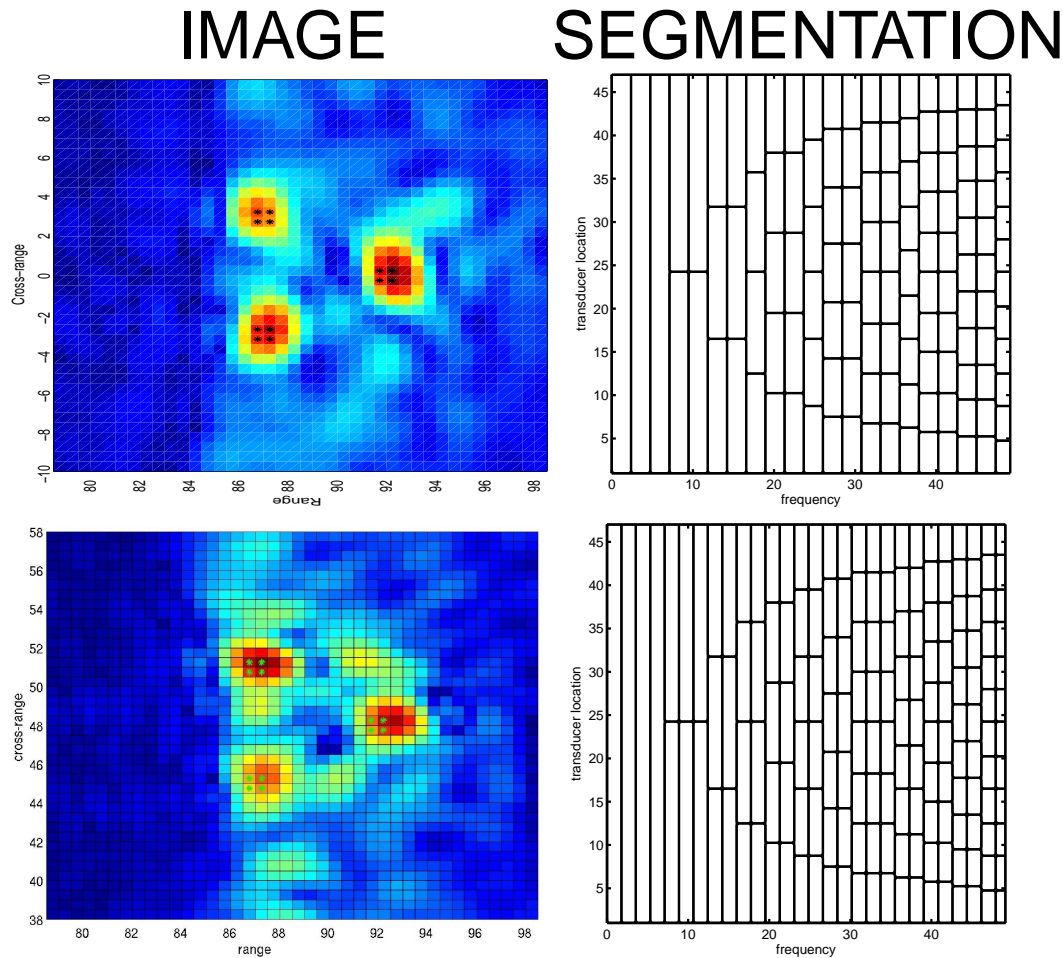
- where \mathcal{N} is a given noisy image, \mathcal{I} is the desired denoised image, $\|\cdot\|_{\text{prox}}$ is a proximity norm, usually $L^2(D)$, and $\|\cdot\|_{\text{reg}}$ is a regularization norm, usually TV.
- We do not have an image \mathcal{N} so there is no proximity norm part. We use instead the L^1 norm of the image which is small, when the image is sparse. We do have however the regularization term.

Adaptive CINT results I



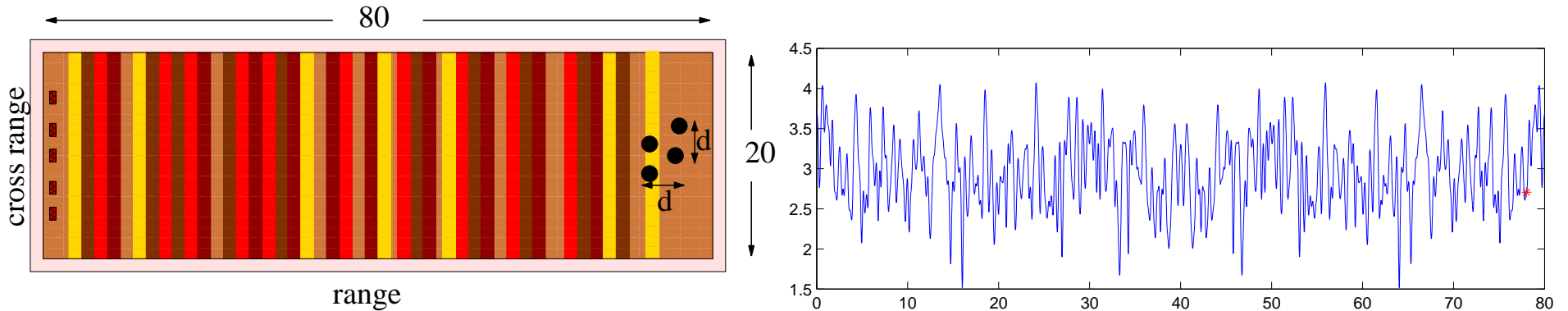
- We use the NOMADm software package (C. Audet, J. Dennis, M. Abramson), that uses a mesh-adaptive direct search method for constrained, nonlinear, mixed variable problems.

Adaptive CINT Results II



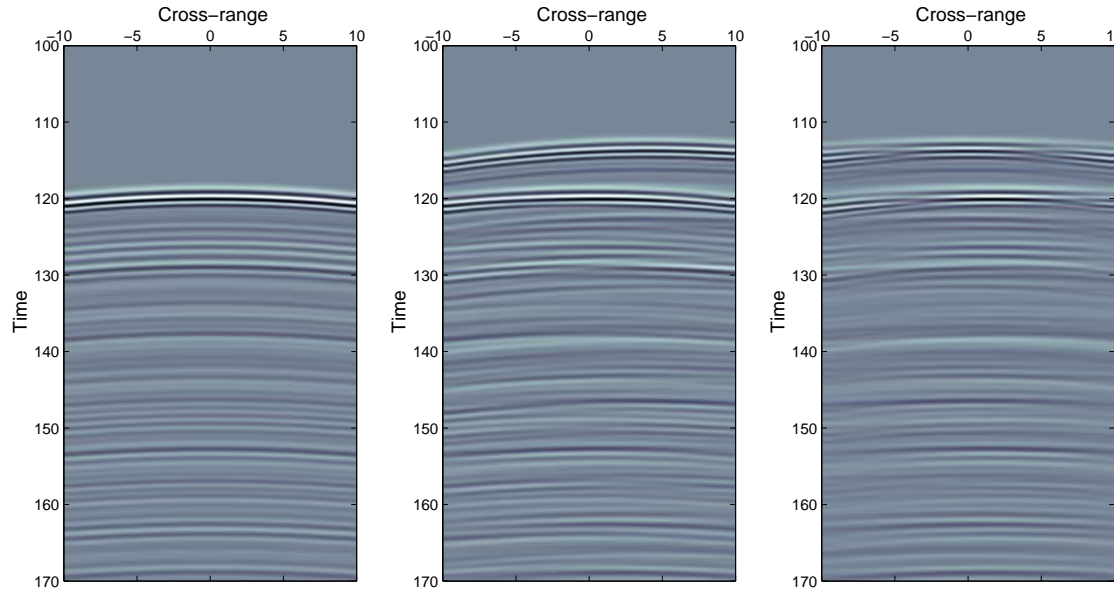
Top: mono-scale, Bottom: multi-scale random medium with standard deviation 3%.

Anisotropic clutter



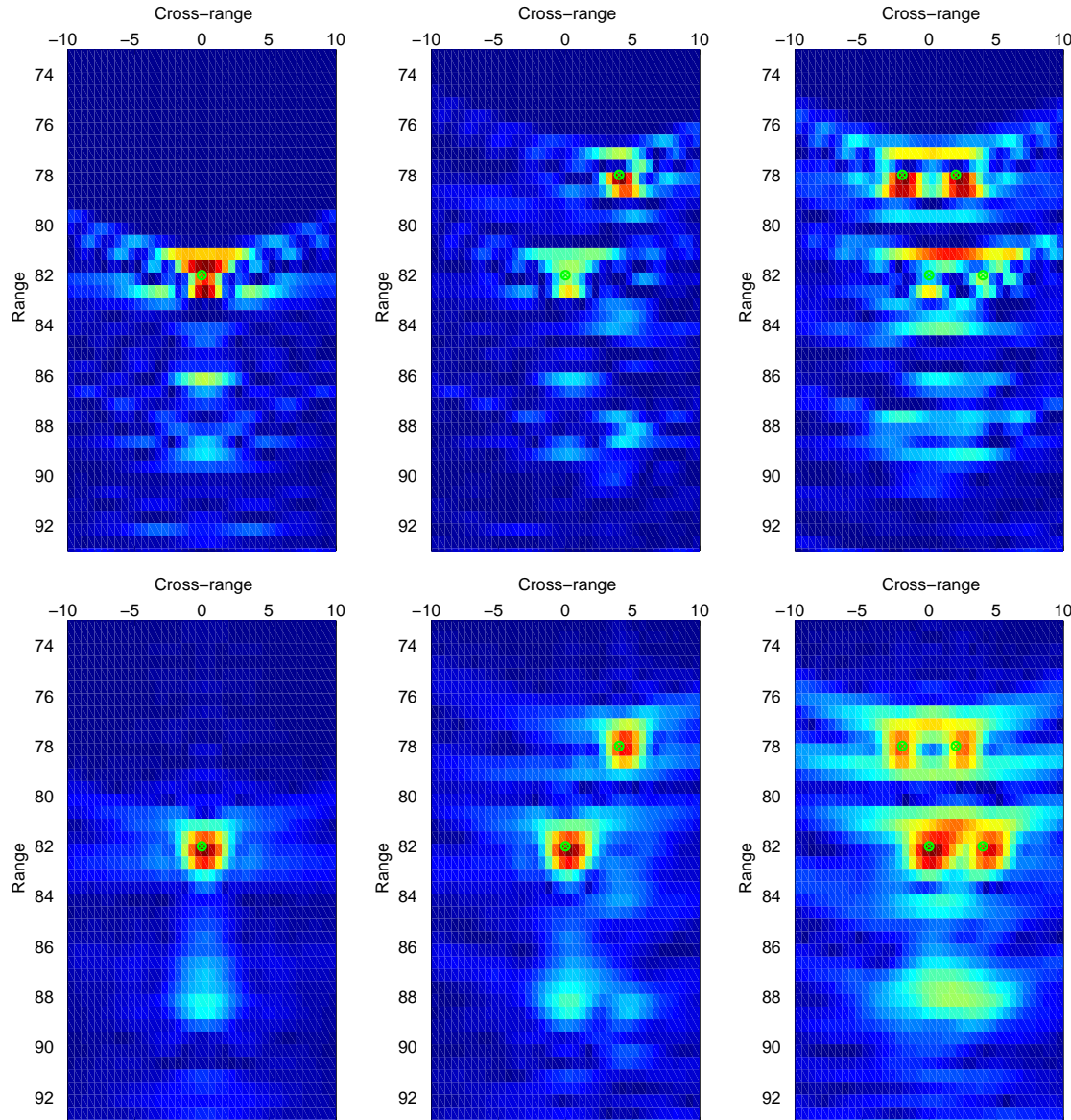
- $c_0 = 3\text{Km/s}$, $B = 0.6 - 1.3 \text{ KHz}$, $\lambda_0 = 3\text{m}$
- $l = \lambda_0/10$, $L = 80\lambda_0 = 800l$
- strong fluctuations std $s = 30\%$,
- In this regime we have pulse stabilization (ODA) and in the limit $\lambda_0/L = \epsilon \rightarrow 0$ KM is stable
- CINT here is obtained using only window $\Psi(t; T_d)$, i.e $\Phi(\kappa; \kappa_d)$ is a δ function

Anisotropic clutter: traces



- The ordinate in the pictures is time scaled by the pulse-width and the abscissa is the array element position in λ_0 .

Anisotropic clutter: KM vs CINT



References

● Finely Layered media

- M. Asch, W. Kohler, G. Papanicolaou, M. Postel, and B. White, Frequency content of randomly scattered signals, *SIAM Rev.*, 33 (1991), pp. 519-625.
- J. F. Clouet and J. P. Fouque, Spreading of a pulse travelling in random media, *Ann. Appl. Probab.*, 4 (1994), pp. 1083-1097.
- J. Chillan and J. P. Fouque, Pressure fields generated by acoustical pulses propagating in randomly layered media, *SIAM J. Appl. Math.*, 58 (1998), pp. 1532-1546.
- J. P. Fouque, S. Garnier, A. Nachbin, and K. Solna, Time-reversal refocusing for point source in randomly layered media, *Wave Motion*, 42 (2005), pp. 238-260.
- J. Garnier, Imaging in randomly layered media by cross-correlating noisy signals, *Multiscale Model. Simul.*, 4 (2005), pp. 610-640.
- L. Borcea, G. Papanicolaou and CT, Coherent interferometry in finely layered random media, *SIAM Journal on Multiscale Modeling and Simulation*, vol 5, (2006), pp. 62 - 83.
- L. Borcea, G. Papanicolaou and CT, Coherent Interferometric Imaging, to appear in "Geophysics", 2006.

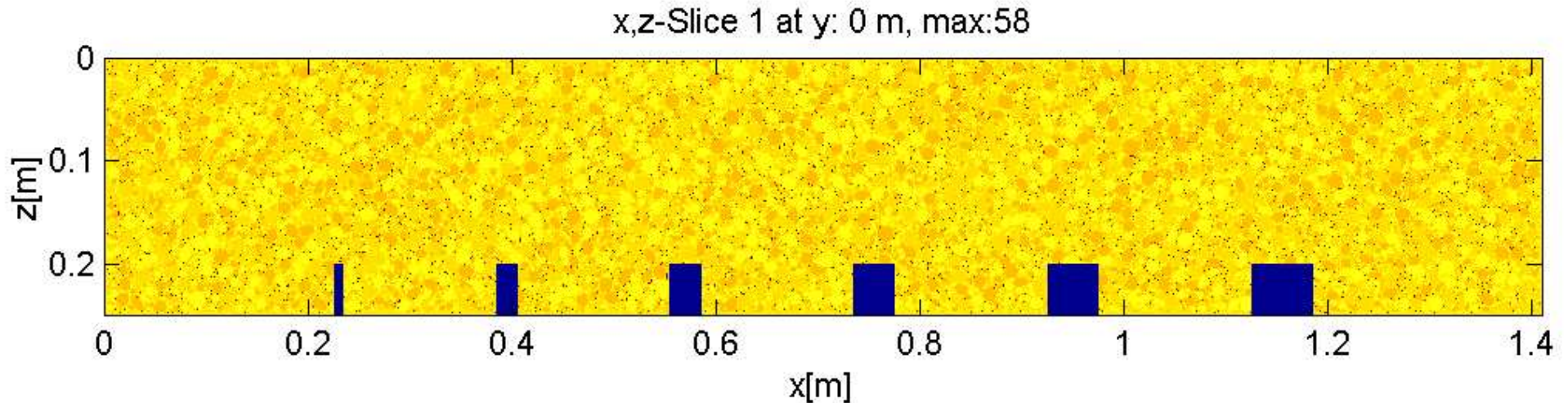
References

● Isotropic clutter

- L. Borcea, G. Papanicolaou and CT, Theory and applications of time reversal and interferometric imaging, *Inverse Problems*, vol 19, (2003), pp. 5139-5164.
- L. Borcea, G. Papanicolaou and CT, Interferometric array imaging in clutter, *Inverse Problems*, vol 21, (2005), pp. 1419-1460.
- L. Borcea, G. Papanicolaou and CT, Adaptive interferometric imaging in clutter and optimal illumination, to appear in *Inverse Problems*, 2006.
- G. Bal, G. Papanicolaou and L. Ryzhik, Self-averaging in time reversal for the parabolic wave equation, *Stochastics and Dynamics*, 2, (2002), pp. 507-531.
- G. Bal and L. Ryzhik, Time reversal and refocusing in random media: *SIAM Journal on Applied Mathematics*, 63 (2003), 1475 -1498.
- G. Papanicolaou, L. Ryzhik and K. Solna, Statistical stability in time reversal, *SIAM J. on Appl. Math.*, 64 (2004), pp. 1133-1155.

Real data: measurements

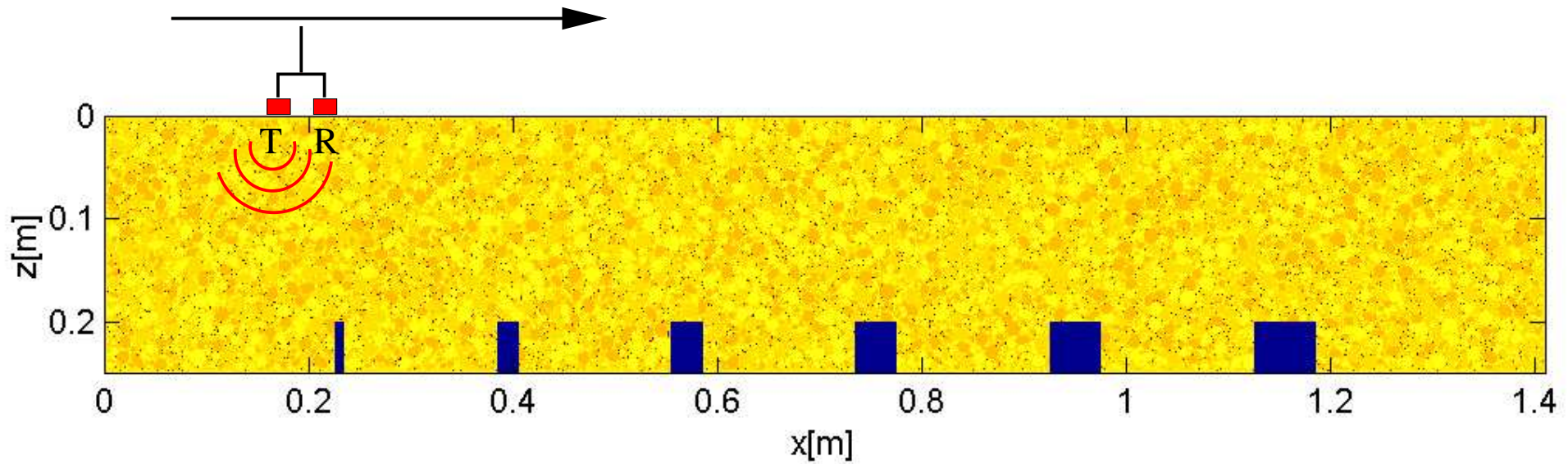
- Concrete structure to be imaged



- data provided by K. Mayer, University of Kassel, Germany.
- simulation in homog. medium: $f_0 = 200\text{KHz}$, $c_0 = 4207\text{m/s}$
- experimental data: $f_0 = 150\text{KHz}$, $c_L = 4150\text{m/s}$
- Transmitter and receiver: Krautgrämer G0,2R

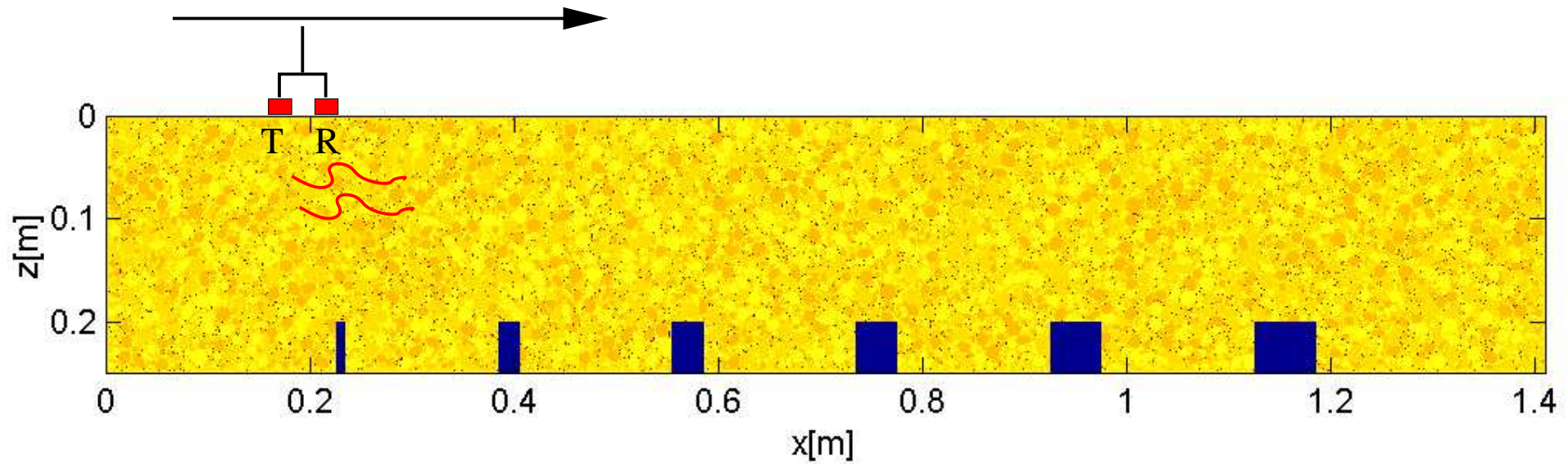
Real data: measurements

- Measurement acquisition geometry



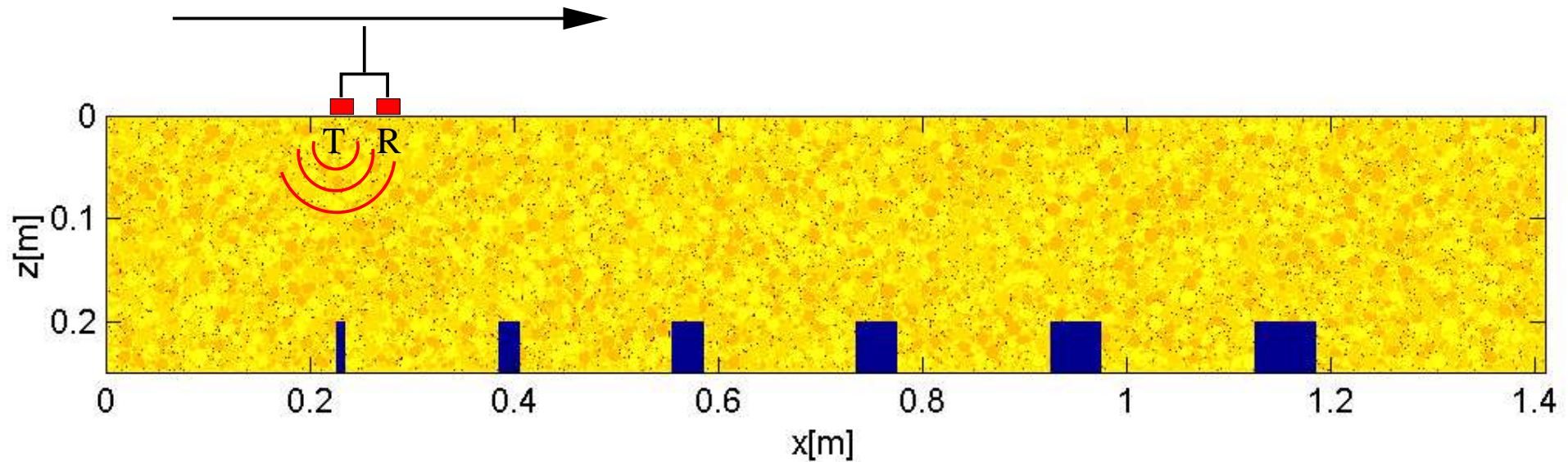
Real data: measurements

- Measurement acquisition geometry



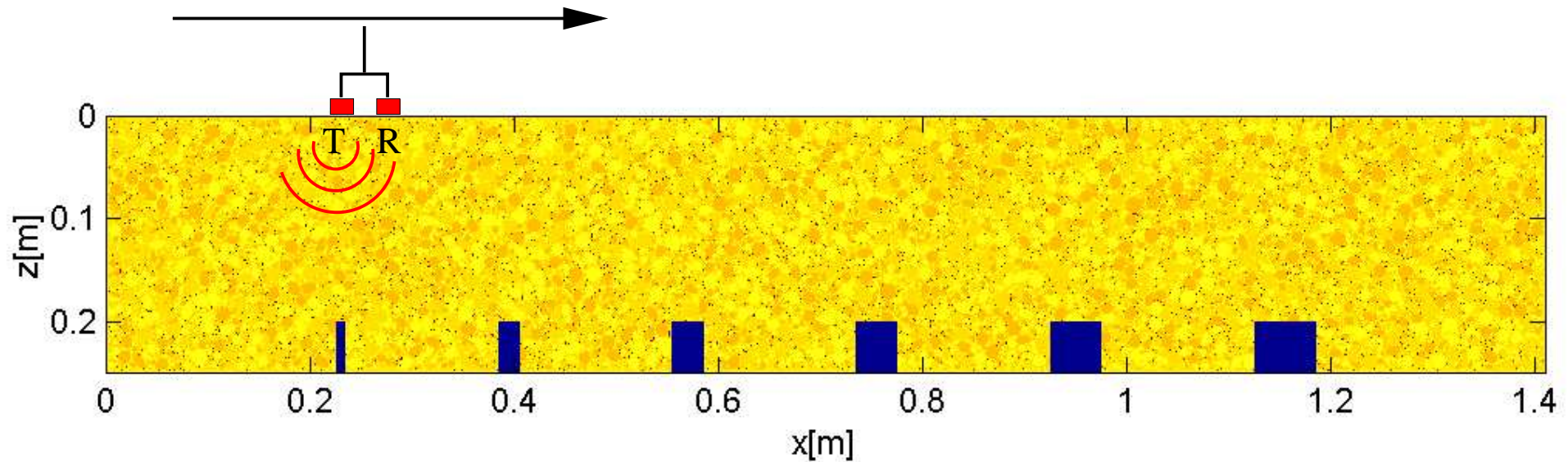
Real data: measurements

- Measurement acquisition geometry

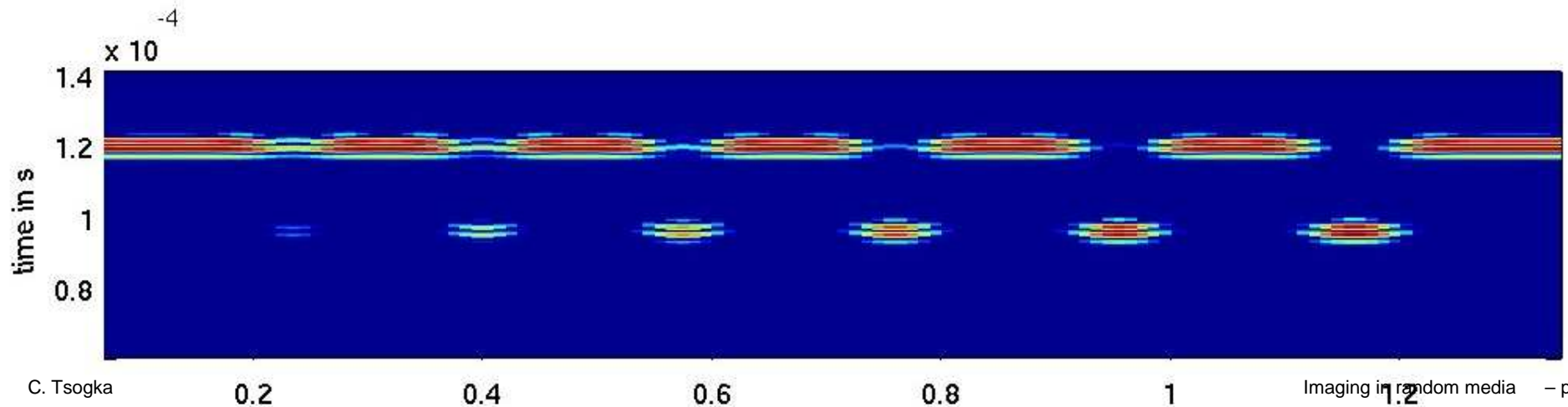


Real data: measurements

- Measurement acquisition geometry



- Simulated data traces in homogeneous structure



The CINT functional

• We rewrite the CINT imaging functional

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^S, \Omega_d, \kappa_d) = \int_B d\omega \int_{|\omega - \omega'| \leq \Omega_d} d\omega' \sum_{\mathbf{x}_m \in a} \sum_{|\mathbf{x}_m - \mathbf{x}_m'| \leq X_d(\omega)}$$

$$\hat{\mathcal{F}}\left(\mathbf{x}_m - \frac{d}{2}, \mathbf{x}_m + \frac{d}{2}, \omega, \mathbf{y}^S\right) \hat{\mathcal{F}}\left(\mathbf{x}_m' - \frac{d}{2}, \mathbf{x}_m' + \frac{d}{2}, \omega', \mathbf{y}^S\right)$$

$$\hat{\mathcal{F}}(\mathbf{x}_s, \mathbf{x}_r, \omega, \mathbf{y}^S) = \hat{P}(\mathbf{x}_s, \mathbf{x}_r, \omega) e^{-i\omega(\tau(\mathbf{x}_s, \mathbf{y}^S) + \tau(\mathbf{x}_r, \mathbf{y}^S))}$$

with

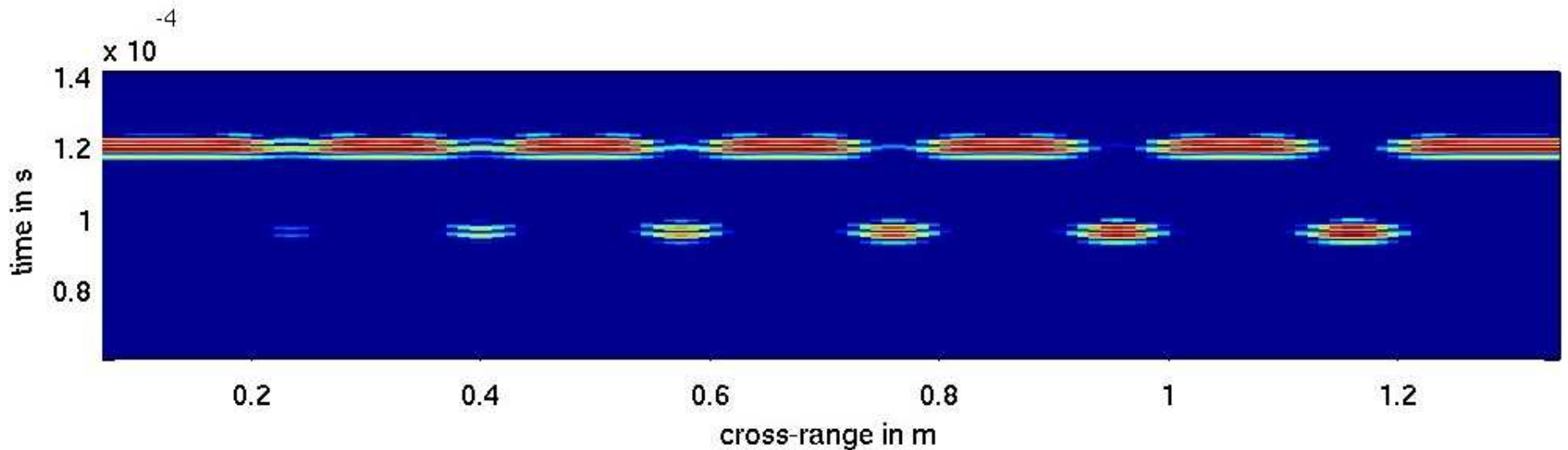
• \mathbf{x}_m : the midpoint moving on the array.

• d : distance between transmitter and receiver (fixed).

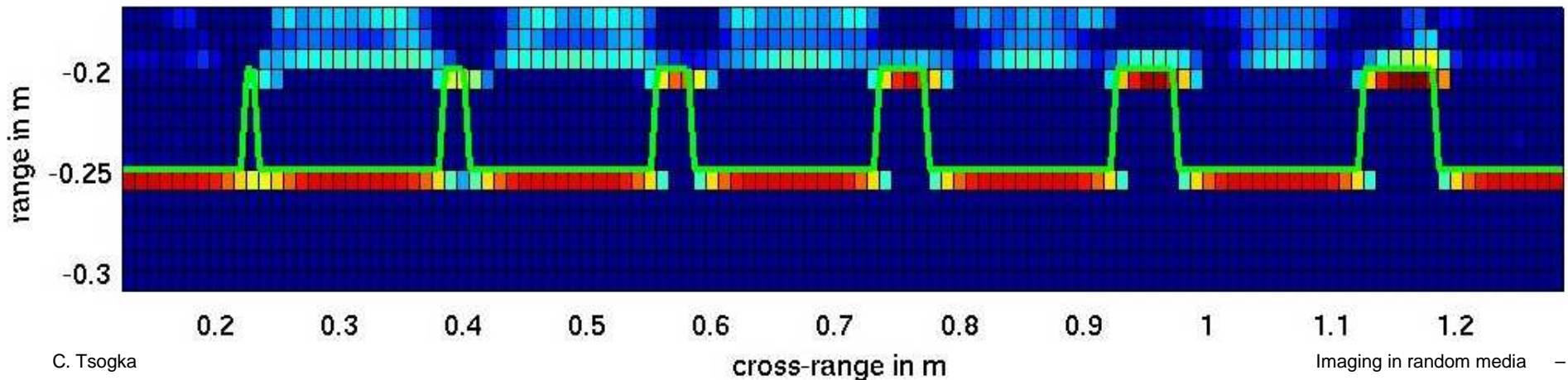
• $\mathbf{x}_s = \mathbf{x}_m - \frac{d}{2}, \mathbf{x}_r = \mathbf{x}_m + \frac{d}{2}$

Adaptive CINT results on real data

- Simulated data traces in homogeneous structure

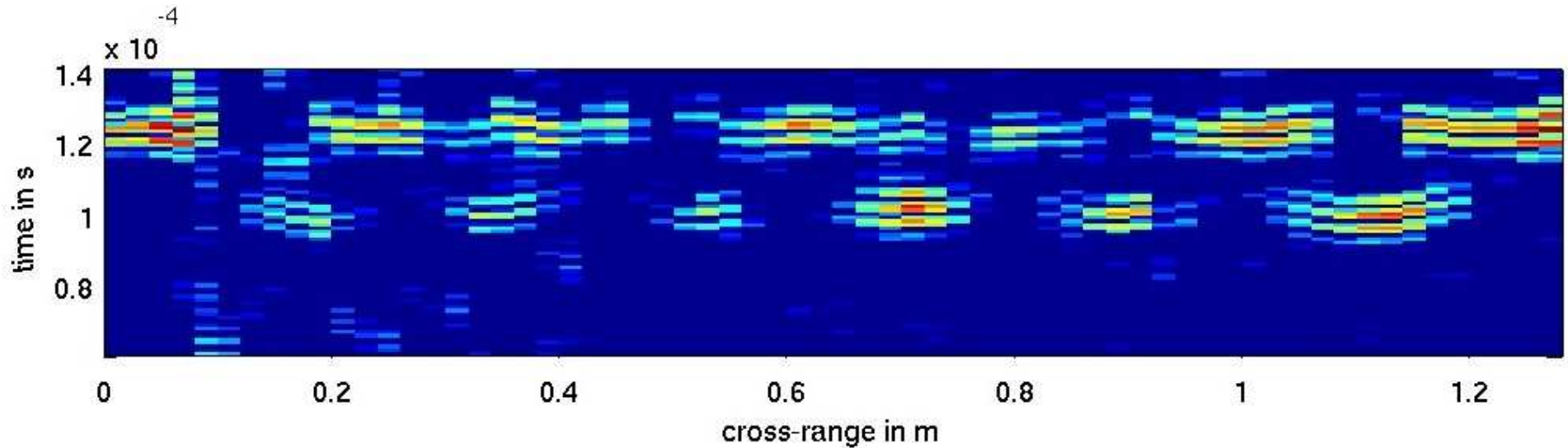


- Kirchhoff migration results

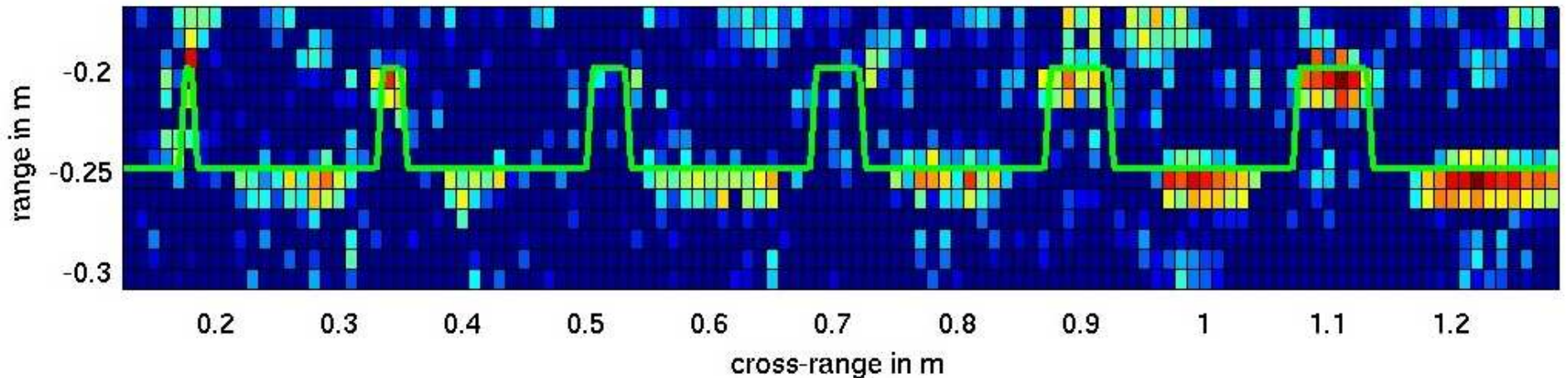


Adaptive CINT results on real data

- Data traces in the concrete structure

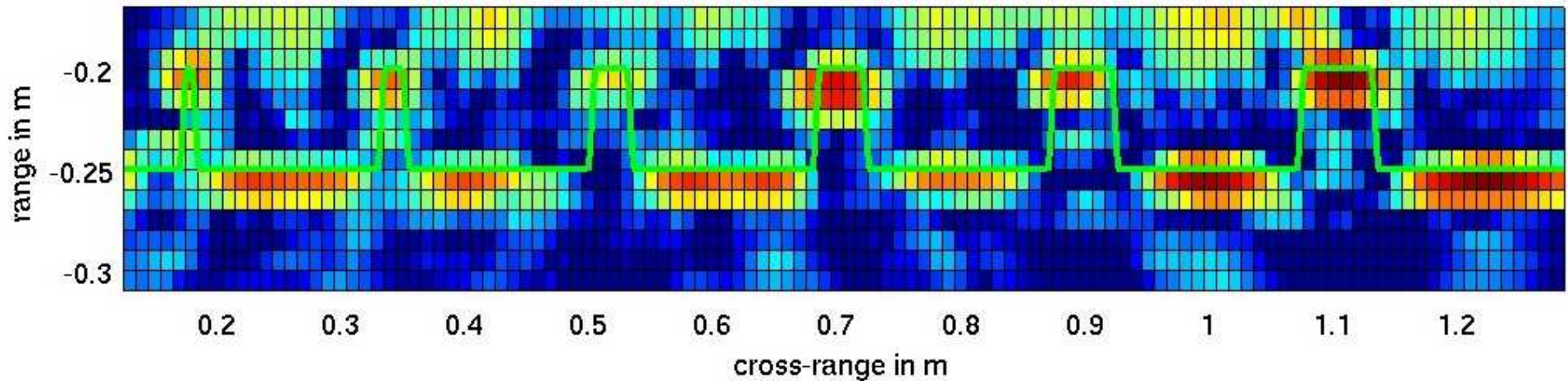


- Kirchhoff migration results



Adaptive CINT results on real data

- Adaptive CINT results



- Kirchhoff migration results

