Wavelet Galerkin FEM

for

Operator Equations with Stochastic Data

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Numerical models in engineering can be solved with high accuracy

if input data are known exactly.

Often, however,

input data are not known exactly

and

accurate numerical solutions are of limited use.

- Mathematical description of uncertainty in input data and solution?
- How to *propagate* data uncertainty through an engineering FEM simulation?
- How to process statistical information in FEM?

Goal:

given statistics of input data, compute (deterministic) solution statistics.

Tool:

Formulation and solution of *Stochastic Partial Differential Equation (SPDE)*

Basic Problem: Operator Equation w. Stochastic Data

Find $u : \Omega \ni \omega \longrightarrow V$ such that

$$Au = f(\cdot, \omega), \qquad f : \Omega \ni \omega \longrightarrow V'$$

References

Perturbation Methods; “First Order Second Moment” (FOSM)

J. B. Keller (1964)

M. Kleiber, T.D. Hien (1992)

CS and R.A. Todor (2003)

Stochastic Galerkin; Wiener Polynomial Chaos (Karhunen-Loève)

R. G. Ghanem, P. D. Spanos (1991)

I. Babuska , J.T. Oden et al. (2001)

G. E. Karniadakis, D. Xiu (2002)

Matthies and Keese (2005)

Outline

Random fields, statistics

$$D \subset \mathbb{R}^d \quad \text{bounded domain,} \quad \Gamma = \partial D = \Gamma_0 \cup \Gamma_1 \quad \text{Lipschitz,}$$
$$(\Omega, \Sigma, P) \quad \text{probability space}$$

Random fields on Γ, D:

X separable Hilbert space. $u(x, \omega)$ *random field* iff

$$u \in L^0(\Omega, X) := \{u(x,\omega) : \quad \Omega \to X| \quad \Omega \ni \omega \to \|u(\cdot,\omega)\|_X \text{ is } P\text{-measurable } \}$$

A random field $u\colon \Omega \to X$ is in $L^1(\Omega, X)$ if $\omega \mapsto \|u(\omega)\|_X$ is integrable so that

$$\|u\|_{L^1(\Omega,X)} := \int_\Omega \|u(\omega)\|_X \, dP(\omega) < \infty$$

In this case the Bochner integral

$$\mathbb{E}u := \int_\Omega u(\omega) dP(\omega) \in X$$

exists and we have

$$\|\mathbb{E}u\|_X \leq \|u\|_{L^1(\Omega,X)} . \qquad (1)$$

$B : X \to Y$ continuous, linear.

$u \in L^k(\Omega, X)$ random field in $X \implies v(\omega) = Bu(\omega) \in L^k(\Omega, Y)$

$$\|Bu\|_{L^k(\Omega,Y)} \leq C \|u\|_{L^k(\Omega,X)}$$

and

$$B \int_\Omega u \, dP(\omega) = \int_\Omega Bu \, dP(\omega).$$

Statistical moments of u: for any $k \in \mathbb{N}$ need k-fold tensor product spaces

$$X^{(k)} = \underbrace{X \otimes \cdots \otimes X}_{k\text{-times}},$$

equipped with natural norm $\| \circ \|_{X^{(k)}}$:

$$\forall u_1, \ldots, u_k \in X \qquad \|u_1 \otimes ... \otimes u_k\|_{X^{(k)}} = \|u_1\|_X ... \|u_k\|_X$$

For $u \in L^k(\Omega, X)$ consider random field

$$u^{(k)} = u(\omega) \otimes \cdots \otimes u(\omega) \in L^1(\Omega, X^{(k)})$$

and

$$\begin{aligned} \left\| u^{(k)} \right\|_{L^1(\Omega,X^{(k)})} &= \int_\Omega \|u(\omega) \otimes \cdots \otimes u(\omega)\|_{X^{(k)}} \, dP(\omega) \\ &= \int_\Omega \|u(\omega)\|_X \cdots \|u(\omega)\|_X \, dP(\omega) = \|u\|^k_{L^k(\Omega,X)} \end{aligned} \tag{2}$$

Define k-th moment (k-point correlation function) $\mathcal{M}^k u$ as expectation of $u \otimes \cdots \otimes u$:

Definition 0

For $u \in L^k(\Omega, X)$ for some integer $k \geq 1$, the k-th moment of $u(\omega)$ is defined by

$$\mathcal{M}^k u = \mathbb{E}[\underbrace{u \otimes ... \otimes u}_{k-times}] = \int_{\omega \in \Omega} \underbrace{u(\omega) \otimes \; ... \otimes u(\omega)}_{k-times} \, dP(\omega) \in X^{(k)} \tag{3}$$

Application: Covariance of $u \in L^2(\Omega, V)$, V separable and reflexive.

$$\mathrm{Cov}[u] = \mathbb{E}\left[(u - \mathbb{E}u) \otimes (u - \mathbb{E}u)\right] \in V \otimes V$$

If u "sufficiently regular":
Covariance:

$$\mathrm{Cov}[u](x, x') = \int_\Omega (u(x,\omega) - \mathbb{E}u(x))(u(x',\omega) - \mathbb{E}u(x'))dP(\omega), \qquad x, x' \in D.$$

k-th Moment (k-point correlation function): if $u \in L^k(\Omega, V)$, then

$$\begin{cases} \mathcal{M}^{(k)} u = \mathbb{E}[u \otimes ... \otimes u] \in V^{(k)} := V \otimes ... \otimes V : \\ \mathcal{M}^{(k)} u(x_1, ..., x_k) := \displaystyle\int_\Omega u(x_1, \omega) \otimes ... \otimes u(x_k, \omega) \, dP(\omega) \end{cases}$$

Stochastic Operator Equation

Given $A : V \to V'$ linear, bounded, $f \in L^1(\Omega, V')$, find $u \in L^1(\Omega, V)$:

$$Au = f$$

Assume ex. $\alpha > 0$ and $T : V \to V'$ compact such that

$$\forall v \in V : \left\langle (A + T)\, v, v \right\rangle \geq \alpha \left\| v \right\|_V^2 \tag{4}$$

and

$$\ker A = \{0\} \tag{5}$$

Proposition 1

Assume (4) and (5). Then for every $f \in L^0(\Omega, V')$ exists a unique $u \in L^0(\Omega, V)$ solution of $Au = f$.

Statistics

Mean Field: if $u \in L^1(\Omega, V)$ $\begin{cases} E_u \in V : \\ E_u(x) := \int_\Omega u(x,\omega)\, dP(\omega) \end{cases}$

Covariance: if $u \in L^2(\Omega, V)$ $\begin{cases} C[u] \in V \otimes V : \\ C[u](x,y) := \int_\Omega (u(x,\omega) - E_u(x))(u(y,\omega) - E_u(y))\, dP(\omega) \end{cases}$

Variance: $(\mathrm{Var} u)(x) = \mathbb{E}[u^2](x) - (\mathbb{E}[u](x))^2 = (\mathcal{M}^{(2)}[u])(x,x) - (\mathbb{E}[u](x))^2$

kth Moment: if $u \in L^k(\Omega, V)$ $\begin{cases} \mathcal{M}^{(k)} u \in V^{(k)} := V \otimes \ldots \otimes V : \\ \mathcal{M}^{(k)} u(x_1, \ldots, x_k) := \int_\Omega u(x_1,\omega) \otimes \ldots \otimes u(x_k,\omega)\, dP(\omega) \end{cases}$

Proposition 2

Assume (4) and (5). Then for every $f \in L^k(\Omega, V')$ holds $u \in L^k(\Omega, V)$.

Example: Stochastic Dirichlet Problem

$D \subset \mathbb{R}^3$ bounded, Lipschitz.

$$\Delta U = 0 \text{ in } D$$

subject to Dirichlet boundary conditions

$$\gamma_0 U = U|_\Gamma = u \text{ on } \Gamma.$$

Given

$$u \in L^k(\Omega, H^{\frac{1}{2}}(\Gamma)), \qquad k \geq 0,$$

ex. unique solution

$$U(x,\omega) \in L^k(\Omega, H^1(D)) \qquad \text{(Sch. \& Todor 2003).}$$

Example: BEM for Stochastic Dirichlet Problem

$$U(x,\omega) = (SL\sigma)(x,\omega) := \int_\Gamma e(x,y)\,\sigma(y,\omega)ds_y.$$

$$V = H^{-1/2}(\Gamma), \quad \sigma(x,\omega) : \Omega \to H^{-1/2}(\Gamma) \quad \text{random flux}$$

Fubini: SL and $\mathcal{M}^{(1)}$ commute. Hence

$$\mathbb{E}[U] = \mathcal{M}^{(1)}[U] = \mathcal{M}^{(1)}[SL\sigma] = SL\left[\mathcal{M}^{(1)}[\sigma]\right] = SL\left[\mathbb{E}[\sigma]\right]$$

where the mean field $\mathbb{E}[\sigma] = \mathcal{M}^{(1)}[\sigma] \in H^{-\frac{1}{2}}(\Gamma)$ satisfies first kind deterministic BIE

$$S\mathbb{E}[\sigma] = \mathbb{E}[u] \in H^{\frac{1}{2}}(\Gamma)\,. \tag{6}$$

Unique Solvability (Nédélec and Planchard (1973)): ex. $\gamma > 0$ such that

$$\forall \sigma \in H^{-1/2}(\Gamma) : \quad \langle \sigma, S\sigma \rangle \geq \gamma \, \|\sigma\|^2_{H^{-1/2}(\Gamma)}$$

Example: BEM for Stochastic Dirichlet Problem

If in the stochastic Dirichlet problem $u \in L^2\big(\Omega, H^{\frac{1}{2}}(\Gamma)\big)$ and $\mathbb{E}[u] = 0$, then $U \in L^2\big(\Omega, H^1(D)\big)$ and

$$C[U] = \mathcal{M}^{(2)}U = \mathcal{M}^{(2)}(SL\psi) = (SL \otimes SL)\mathcal{M}^{(2)}\psi = \int\limits_\Gamma \int\limits_\Gamma e(x,z)\, e(y,w)\, C[\sigma](z,w) ds_z\, ds_w\,,$$

where

$$C[\sigma] \in H^{-\frac{1}{2},-\frac{1}{2}}(\Gamma \times \Gamma) := H^{-\frac{1}{2}}(\Gamma) \otimes H^{-\frac{1}{2}}(\Gamma)$$

satisfies the first kind BIE

$$(S \otimes S)\, C[\sigma] = C[u] \in H^{\frac{1}{2},\frac{1}{2}}(\Gamma \times \Gamma)\,.$$

Solvability:

$$\forall C[\sigma] \in H^{-\frac{1}{2},-\frac{1}{2}}(\Gamma \times \Gamma): \quad \langle (S \otimes S)C[\sigma], C[\sigma]\rangle \geq c_S^2 \|C[\sigma]\|^2_{H^{-\frac{1}{2},-\frac{1}{2}}(\Gamma\times\Gamma)}$$

Goal of Computation

For the operator equation

$$Au = f$$

with $f \in L^k(\Omega, V)$,

$$\text{given } \mathcal{M}_f^{(k)}, \quad \text{find } \mathcal{M}_u^{(k)}.$$

Approaches:

- Monte-Carlo Galerkin FEM ("Collocation in ω"): dense and sparse
- Sparse Wavelet FEM for deterministic approximation of $\mathcal{M}^{(k)}$

Monte Carlo - I

Given data ensemble

$$\{f(\omega_j), \quad j = 1, ..., M\} \subset V'$$

generate (in parallel) solution ensemble

$$\{u(\omega_j), \quad j = 1, ..., M\} \subset V$$

Theorem 3

Assume (4) and (5) and that $f \in L^{2k}(\Omega, V')$.
Estimate $\mathcal{M}^{(k)}u$ by the k-th moment of ensemble $\{u(\omega_j) : j = 1, ..., M\}$, i.e. by

$$\bar{E}^M_{\mathcal{M}^{(k)}u} := \overline{u \otimes \cdots \otimes u}^M = \frac{1}{M} \sum_{j=1}^{M} u(\omega_j) \otimes ... \otimes u(\omega_j) \in V^{(k)}.$$

Then ex. $C(k) > 0$ such that for every $M \geq 1$ and every $0 < \varepsilon < 1$ holds

$$P\left(\|\mathcal{M}^{(k)}u \ - \bar{E}^M_{\mathcal{M}^{(k)}u}\|_{V\otimes...\otimes V} \leq C \frac{\|\mathcal{M}^{2k}(f)\|^{1/2}_{V'^{(2k)}}}{\sqrt{\varepsilon M}} \right) \geq 1 - \varepsilon \tag{7}$$

Monte Carlo - II

Lemma (Law of iterated logarithm in Hilbert spaces):

V separable Hilbert and $X \in L^2(\Omega, V)$. Then

$$\limsup_{M\to\infty} \frac{\left\| \overline{X}^M - E(X) \right\|_V}{(2M^{-1}\log\log M)^{1/2}} \leq \|X - E(X)\|_{L^2(\Omega,V)} \quad \text{with probability 1.} \tag{8}$$

Proof: Classical law of iterated logarithm: for real valued $Y(\omega)$ holds

$$\limsup_{M\to\infty} \frac{\left| \overline{Y}^M - E(Y) \right|^2}{2M^{-1}\log\log M} = \mathrm{Var}Y \quad \text{with probability 1.}$$

Let $Z := X - E(X)$. V separable $\Rightarrow$ w.l.o.g $V = \ell^2 = \mathrm{span}\{e_j\}_{j=1}^\infty$ and $Y := (e^j, Z) = Z_j \in \mathbb{R}$. Apply (8) with

$$\mathrm{VarY} = (e^j \otimes e^j, \mathcal{M}^2 Z) = (\mathcal{M}^2 Z)_{j,j}.$$

Add estimates for $j = 1, 2, \ldots$ and obtain

$$\limsup_{M\to\infty} \frac{\sum_{j=1}^\infty |Z_j|^2}{2M^{-1}\log\log M} \leq \sum_{j=1}^\infty (\mathcal{M}^2 Z)_{j,j} \quad \text{with probability 1.}$$

Monte Carlo - II

Application: P-a.s. convergence of MCM (*Semidiscrete Case !*)

Theorem 4

Let $f \in L^{2k}(\Omega, V')$. Then

$$\limsup_{M\to\infty} \frac{\left\| \bar{E}^M_{\mathcal{M}^k u} - \mathcal{M}^k u \right\|_{V^{(k)}}}{(2M^{-1}\log\log M)^{1/2}} \leq C \, \|f\|^k_{L^{2k}(\Omega,V')} \quad \text{with probability 1.}$$

Monte Carlo - III
MCM - convergence in the absence of 2nd Moments

Theorem 5

Let $k \geq 1$ and assume

$$f \in L^{\alpha k}(\Omega, V') \quad \text{for some} \quad \alpha \in (1, 2].$$

Then ex. C such that for every $M \geq 1$ and every $0 < \epsilon < 1$

$$P\left(\|\bar{E}^M_{\mathcal{M}^k u} - \mathcal{M}^k u\|_{V^{(k)}} \leq C \frac{\|f\|^k_{L^{\alpha k}(\Omega, V')}}{\varepsilon^{1/\alpha} M^{1-1/\alpha}} \right) \geq 1 - \epsilon \tag{9}$$

So far: MCM assuming that $Au = f$ solved exactly ("Semidiscrete MCM").

Next: Galerkin FEM in V.

Galerkin FEM

Dense sequence of subspaces:

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\ell \subset V_{\ell+1} \subset \dots V$$

Galerkin FEM: given $f \in L^k(\Omega, V')$, find

$$\boxed{u^L(\omega) \in L^k(\Omega, V^L) \text{ such that } \quad \langle v^L, Au^L(\omega)\rangle = \langle v^L, f(\omega)\rangle \qquad \forall v^L \in V_L}$$

Galerkin Projection: $G_L : V \to V_L$ defined by

$$\forall v \in V_L : \quad \langle AG_L u, v\rangle = \langle f, v\rangle$$

is stable: ex. $L_0 > 0$ s.t.

$$\forall L \geq L_0 : \qquad \|G_L u\|_V \leq C\|u\|_V$$

and converges quasioptimally:

$$\forall L \geq L_0 \quad \forall v \in V_L : \qquad \|u(\omega) - u_L(\omega)\|_V \leq C\|u(\omega) - v\|_V \quad P - \text{a.e.}\, \omega \in \Omega.$$

Convergence Rates

Smoothness Spaces:

$$\{X_s\}_{s\geq 0}, \quad X_0 = V, \quad X_s \subseteq V, \qquad \{Y_s\}_{s\geq 0}, \quad Y_0 = V', \quad Y_s \subseteq V'$$

Regularity:

$$A^{-1} : Y_s \ni f \to u \in X_s, \quad s \geq 0.$$

Convergence Rate:

$$\|u(\omega) - u_L(\omega)\|_V \leq C\Phi(s, N_\ell)\, \|u\|_{X_s} \quad \text{where} \quad \Phi(s, N_\ell) := \sup_{v \in X_s} \inf_{v_\ell \in V_\ell} \frac{\|v - v_\ell\|_V}{\|v\|_{X_s}} \quad .$$

MC Galerkin: given $\{f(\omega_j) : j = 1, ..., M\}$, compute $\{u_L(\omega_j) : j = 1, ..., M\}$ and

$$\bar{E}^{M,L}_{\mathcal{M}^k u} := \frac{1}{M} \sum_{j=1}^{M} \underbrace{u_L(\omega_j) \otimes ... \otimes u_L(\omega_j)}_{k-\text{times}} \in V_L^{(k)}.$$

Work:

$$O(N_L^k) \qquad \text{where} \quad N_L = \text{dim} V_L.$$

Wavelet FEM (Cohen, Dahmen, Kunoth, Schneider, ...)

Wavelet Scale:

$$W_0 := V_0, \qquad V_\ell = V_{\ell-1} \oplus W_\ell,\ \ell = 1, 2, \dots,$$

Sparse Tensor Product Space (Smol'yak, Teml'yakov, Zenger, Griebel,...):

$$\widehat{V}_L^{(k)} = \sum_{\substack{\vec{\ell} \in \mathbb{N}_0^k \\ |\vec{\ell}| \leq L}} W_{\ell_1} \otimes W_{\ell_2} \otimes \cdots \otimes W_{\ell_k}\,.$$

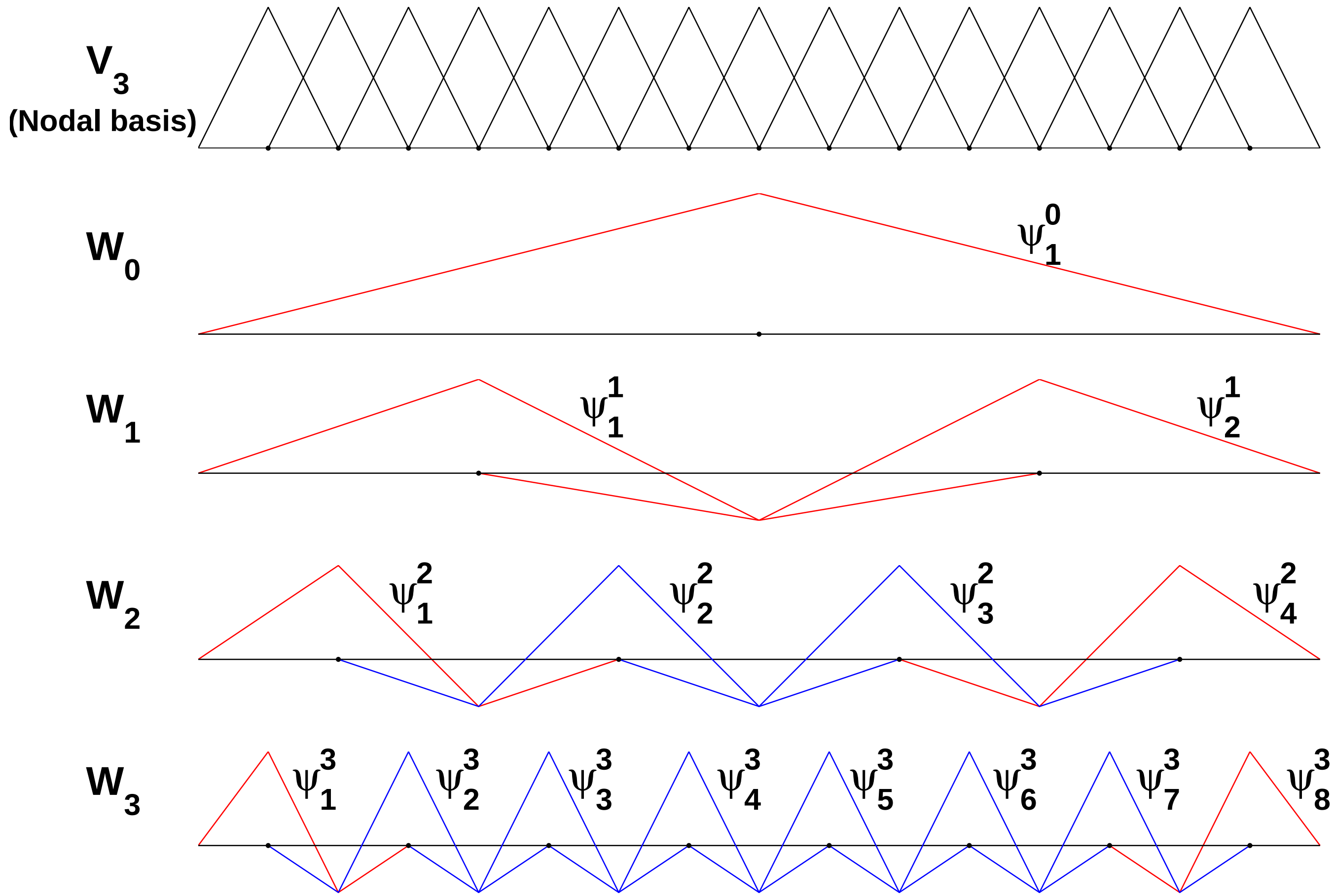
Sparse Projection (quasi-interpolation):

$$\widehat{P}_L^{(k)} : V^{(k)} \to \hat{V}_L^{(k)} \text{ given by } (\widehat{P}_L^{(k)} v)(x) := \sum_{\substack{0 \leq \ell_1 + \cdots + \ell_k \leq L \\ 1 \leq j_\nu \leq n_{\ell_\nu}, \nu = 1, \dots, k}} v_{j_1 \dots j_k}^{\ell_1 \dots \ell_j}\, \psi_{j_1}^{\ell_1}(x_1) \dots \psi_k^{\ell_k}(x_k)$$

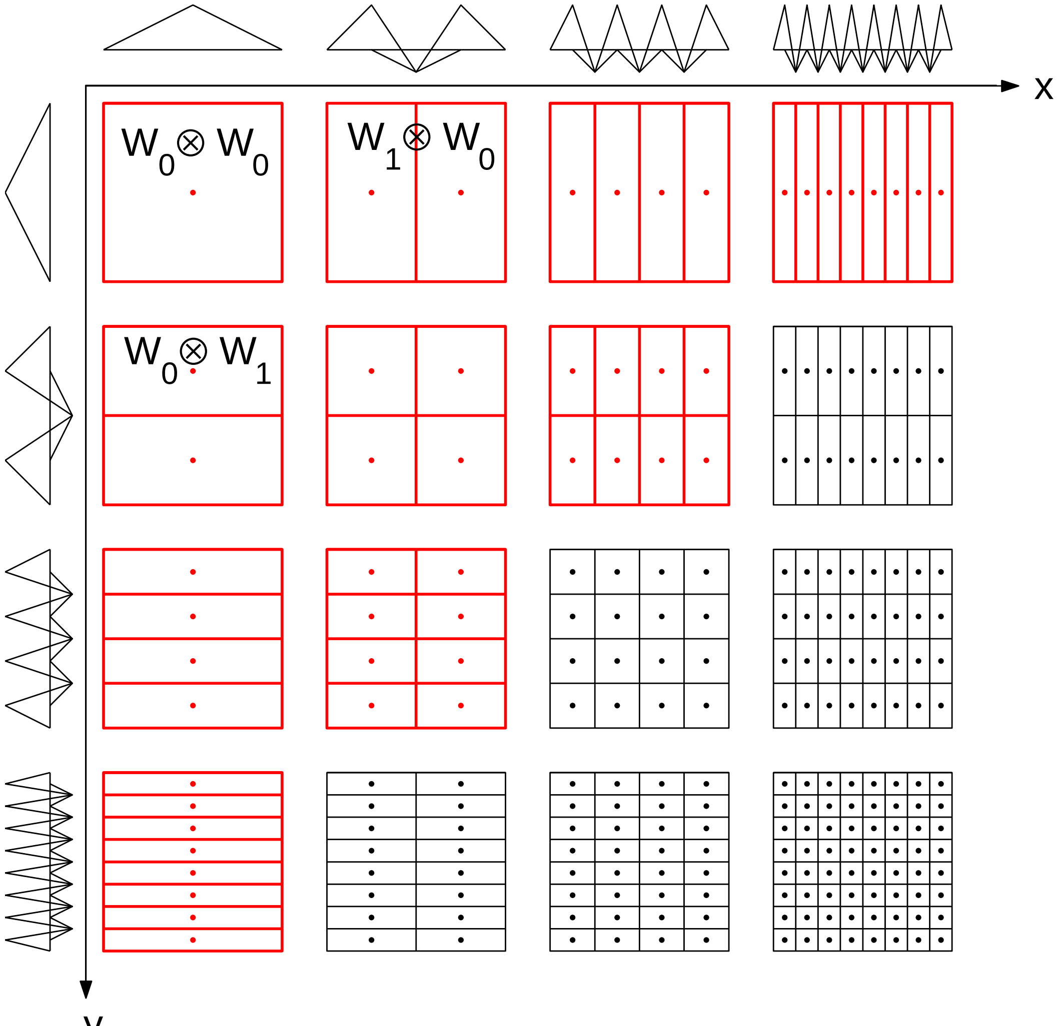
or

$$\widehat{P}_L^{(k)} = \sum_{0 \leq \ell_1 + \cdots + \ell_k \leq L} Q_{\ell_1} \otimes \cdots \otimes Q_{\ell_k} \quad \text{where} \quad Q_\ell := P_\ell - P_{\ell-1}, \ell = 0, 1, \dots \text{ and } P_{-1} := 0.$$

Biorthogonal Spline Wavelets in $1-d$, degree $p=1$.



Sparse Tensor Product Space
(Zenger 1990, Griebel & Bungartz Acta Numerica 2004)



Monte Carlo IV – Sparse Monte Carlo FEM

Sparse Tensor Product MC estimate of $\mathcal{M}^k u$:

$$\hat{E}^{M,L}_{\mathcal{M}^k u} := \frac{1}{M} \sum_{j=1}^{M} \widehat{P}_L^{(k)} \left[u_L(\omega_j) \otimes \ldots \otimes u_L(\omega_j) \right] \in V_L^{(k)}.$$

Work:

$$M \ \times \ O(N_L (\log_2 N_L)^{k-1}) \quad \text{operations and} \quad N_L (\log_2 N_L)^{k-1} \quad \text{memory}$$

Theorem 6

Assume $0 < \alpha \le 1$ and

$$f \in L^k(\Omega, Y_s) \cap L^{\alpha k}(\Omega, V') \text{ for some } 0 \le s < s_0.$$

Then

$$\mathcal{M}^k u \in X_s \otimes \ldots \otimes X_s =: X_s^{(k)}$$

and there is $C(k) > 0$ such that for all $M \ge 1$, $L \ge L_0$ and all $0 < \varepsilon < 1$ holds

$$P \left(\| \hat{E}^{M,L}_{\mathcal{M}^k u} - \mathcal{M}^k u \|_{V^{(k)}} < \lambda \right) \ge 1 - \varepsilon$$

with $\lambda = C(k) \left[\Phi(s, N_L) (\log N_L)^{(k-1)/2} \, \|f\|^k_{L^k(\Omega, Y_s)} + \varepsilon^{-1/\alpha} M^{-(1-1/\alpha)} \, \|f\|^k_{L^{\alpha k}(\Omega, V')} \right].$

Sparse Tensor Product FEM

Idea:

Compute $\mathcal{M}^k u$ directly, *without* MC

Proposition 7

Assume A satisfies (4), (5) and $f \in L^k(\Omega, V')$ for $k > 1$.

Then

$$(A \otimes ... \otimes A) Z = \mathcal{M}^k f \,, \tag{10}$$

has a unique solution $Z \in V^{(k)}$ and

$$Z = \mathcal{M}^k u.$$

For $f \in L^k(\Omega, Y_s)$, $s > 0$, holds

$$\|\mathcal{M}^k u\|_{X_s \otimes ... \otimes X_s} \leq C_{k,s} \, \|\mathcal{M}^k f\|_{Y_s \otimes ... \otimes Y_s}, \qquad 0 \leq s < s_0, \quad k \geq 1$$

Regularity of $\mathcal{M}^k u$ in spaces of mixed highest derivative!

Sparse Galerkin FEM

Since A may be indefinite, use

$$\hat{V}^{(k)}_{L,L_0} := \hat{V}^{(k)}_{L+L_0} \cap V^{(k)}_L$$

instead of $\hat{V}^{(k)}_L$ where L_0 is fixed as $L \to \infty$.

$$\text{find} \quad \widehat{Z}_L \in \widehat{V}^{(k)}_{L+L_0} \quad \text{such that} \quad \left\langle A^{(k)}\, \widehat{Z}_L, v \right\rangle = \left\langle \mathcal{M}^k f, v \right\rangle \quad \forall v \in \widehat{V}^{(k)}_{L+L_0}\,. \tag{11}$$

N.B. that

$$\hat{V}^{(k)}_L \subset \hat{V}^{(k)}_{L,L_0} \subset \hat{V}^{(k)}_{L+L_0}.$$

Assume (4), (5) with

$$T : V \to Y_\delta \quad \text{continuously for some} \quad \delta > 0$$

Assume also the approximation property and $f \in L^k(\Omega, Y_s)$, $s > 0$, $k \geq 1$.

Then there exists L_0 and $c_S > 0$ such that for all $L \geq L_0$

$$\inf_{0 \neq u \in \widehat{V}^{(k)}_{L,L_0}} \; \sup_{0 \neq v \in \widehat{V}^{(k)}_{L,L_0}} \; \frac{\langle A^{(k)} u, v \rangle}{\|u\|_{V^{(k)}}\, \|v\|_{V^{(k)}}} \geq \frac{1}{c_S} > 0\,. \tag{12}$$

In the case $T = 0$ this holds with $L_0 = 0$.

Theorem 8

Then for all $L \geq kL_0$ sparse Galerkin approximation $\widehat{Z}_L$ of $\mathcal{M}^k u$ is uniquely defined and

$$\|\mathcal{M}^k u - \widehat{Z}_L\|_{V\otimes\ldots\otimes V} \leq C(k)\Phi(s, N_L)(\log N_L)^{(k-1)/2}\|f\|_{L^k(\Omega,Y_s)}, \qquad 0 \leq s < s_0.$$

$\widehat{Z}_L$ can be computed with $O(N_L(\log N_L)^m)$ work and memory.

Note: Tensor product Galerkin FEM gives

$$\|\mathcal{M}^k u - Z_L\|_{V\otimes\ldots\otimes V} \leq C(k)\Phi(s, N_L)^{1/k}\|f\|_{L^k(\Omega,Y_s)}, \qquad 0 \leq s < s_0$$

in $O(N_L)$ memory and work.

Conclusions

- Monte-Carlo Galerkin FEM: framework, convergence analysis
- Sparse Galerkin FEM: regularity in anisotropic spaces; sparse tensor product spaces
- Given data statistics, get solution statistics by deterministic computation
- trade stochasticity and MC for high-dimensionality + deterministic FEM
- Use sparse tensor products of wavelet spaces to avoid $O(N_L^k)$ complexity
- Fast Matrix Vector Multiplication (Sch. & Todor: Numer. Math. 2003)
- a-priori and a-posteriori error estimates, adaptivity
  $\rightarrow$ framework of Cohen, Dahmen, DeVore in tensor product Besov spaces (Nitsche 2004)

  $$\mathcal{M}^k(u) \in B_q^\alpha(L_q(D)) \otimes_q \ldots \otimes_q B_q^\alpha(L_q(D))$$

  for arbitrarily large α with

  $$q = [\alpha/2 + 1/2]^{-1} < 1 \quad \text{indep. of} \quad k.$$

- Problems with stochastic boundary Γ (Harbrecht, R. Schneider, and CS. 2006)
- Problem with stochastic coefficients (Todor & CS. 2006)