Monte-Carlo Methods for kinetic equations

Boltzmann eq., Vlasov eq., V.F.-P.

Coupling with other models $\Rightarrow$ time splitting

1 Example of Kinetic equation :VF-P

■ Density of particle $f = f(t, x, v)$

Set of velocity R^3 provided with dv s.t. $\int_{\mathcal{V}} dv = \mathbf{1}$, $\mathbf{E}$ electric field

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial x} + \mathbf{E}.\frac{\partial f}{\partial \mathbf{v}} - \mathcal{L}f &= 0, \\ f(0, .) &= G \end{aligned}$$

here $\mathcal{L}$ is conservative (collision operator with the electrons or the ions of a plasma)

$$\mathcal{L}f(\mathbf{v}) = N_e \frac{\partial}{\partial \mathbf{v}}.\left(O(\mathbf{v}-\mathbf{U}).\frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}) \right) + N_e \frac{\partial}{\partial \mathbf{v}}.\left(\mathbf{r}(\mathbf{v}-\mathbf{U}) f(\mathbf{v})\right),$$

where the vector $\mathbf{r}$ and the tensor O are ($\mathbf{e}$ arbitrary unit v.)

$$\mathbf{r}(\mathbf{w}) = \frac{1}{m_e}\frac{\mathbf{w}}{|\mathbf{w}|}\int \frac{\mathbf{y}.\mathbf{e}}{|\mathbf{y}|^3} G(\mathbf{y}-\mathbf{e}|\mathbf{w}|) d\mathbf{y}, \qquad O(\mathbf{w}) = \frac{1}{2m_s}\int \left(\frac{\mathbf{y}\mathbf{y}}{|\mathbf{y}|^3} - \frac{\mathbf{1}}{|\mathbf{y}|}\right) G(\mathbf{y}-\mathbf{w}) d\mathbf{y}, \qquad G(.) \text{ Gaussienne}$$

$$\mathcal{L}f(\mathbf{v}) = N_e \frac{\partial}{\partial \mathbf{v}} \int \left(\frac{(\mathbf{v}_e-\mathbf{v})(\mathbf{v}_e-\mathbf{v})}{|\mathbf{v}_e-\mathbf{v}|^3} - \frac{\mathbf{1}}{|\mathbf{v}_e-\mathbf{v}|} \right) \left(\frac{1}{2m_s} G(\mathbf{v}_e-\mathbf{U}) \frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}) - \frac{1}{2m_e} f(\mathbf{v}) \frac{\partial}{\partial \mathbf{v}_e} G(\mathbf{v}_e-\mathbf{U}) \right) d\mathbf{v}$$

2 Probabilist interpretation

$$f \longmapsto v.\frac{\partial}{\partial x} f + \mathbf{E}.\frac{\partial f}{\partial v} - \mathcal{L}^* f \qquad \text{is associated to the processus } X(t),\ V(t) \tag{1}$$

■ Simple case. $v.\frac{\partial}{\partial x} u + \frac{\partial u}{\partial v}.(\mathbf{F} u)$ associated with

$$\frac{\partial}{\partial t} X(t) = V(t), \qquad \frac{\partial}{\partial t} V(t) = \mathbf{F}(X(t), V(t))$$

■ VFP. Denote $\mathbf{s}(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}}.O(\mathbf{v})$, we get

$$\frac{1}{N_e}\mathcal{L}^* f(\mathbf{v}) = -\mathbf{r}(\mathbf{v}-\mathbf{U})\frac{\partial f}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}}.\left(O(\mathbf{v}-\mathbf{U}).\frac{\partial f}{\partial \mathbf{v}}\right) = -\left(\mathbf{r}(\mathbf{v}-\mathbf{U}) + \mathbf{s}(\mathbf{v}-\mathbf{U})\right)\frac{\partial f}{\partial \mathbf{v}} + O(\mathbf{v}-\mathbf{U}) : \frac{\partial^2 f}{\partial \mathbf{v} \partial \mathbf{v}}$$

Simplify $O(\mathbf{v}-\mathbf{U}) = O(\mathbf{x})$.

The Markov process (X_t, V_t) associated to (1)

$$\frac{d}{dt}X_t = V_t, \tag{2}$$
$$dV_t = dt\,(\mathbf{E} - N_e(\mathbf{r}+\mathbf{s})(V_t - \mathbf{U}(X_t))) + \overline{\overline{\sigma_e}}(X_t)dB_t. \tag{3}$$

where B_t states for the Bronwian motion and $\overline{\overline{\sigma_e}}$ is the square root of $2O_e(\mathbf{x})$.

Consider a particle with velocity V_t on a small time intervall δt, then

$$V_{t+\delta t} = V_t - \delta t N_e(\mathbf{r}+\mathbf{s})(V_t - \mathbf{U}(X_t)) + \mathbf{b}(\delta t)$$

$\mathbf{b}(\delta t)$ is a random vector whose expectation is zero and whose variance is s.t.

$$E[\mathbf{b}(\delta t)\mathbf{b}(\delta t)/(X_t, V_t)] = 2O(X_t)\delta t$$

Proposition (probabilist interpretation). For any test fonction $\varphi \in C_b(D \times V)$,

$$\ll f(t,.)\varphi \gg = \int\int_{\mathcal{DV}} f(t,x,v)\varphi(x,v)\,dx\,dv =$$
$$= E_g\left[\varphi(X(t),V(t))\right]$$

3 Principle of the numerical method

■Generation of N processes, ind. equid. : $(X_p, V_p), p = 1, ..., N$.

■ At each time step (small enough)

1. Track the particles according to a free fly (without coll.)

$$dX_t/dt = V_t$$

2. Acceleration

$$dV_t = dt\,(\mathbf{E} - N_e(\mathbf{r}+\mathbf{s})(V_t - \mathbf{U}(X_t)))$$

$$\begin{aligned} \text{e.g. } V_t^0 &= V_t + \delta t \mathbf{E}(X_t) \\ V_t^+ - \mathbf{U}(X_t) &= \left(V_t^0 - \mathbf{U}(X_t)\right) \frac{1 - N_e|\mathbf{r}+\mathbf{s}|\delta t/2}{1 + N_e|\mathbf{r}+\mathbf{s}|\delta t/2} \end{aligned}$$

3. Diffusion

$$V_t^+ \hookrightarrow V_t^+ + \mathbf{b}(\delta t)$$

Mean value of $\mathbf{b}(\delta t) = 0$

$$E[\mathbf{b}(\delta t)\mathbf{b}(\delta t)/(X_t, V_t)] = 2O(V_t - \mathbf{U}(X_t))\delta t$$

Remark. Momentum deposition :

$$\mathbf{R} = -N_e \int (\mathbf{s}(\mathbf{v}-\mathbf{U}) + \mathbf{r}(\mathbf{v}-\mathbf{U}))\, f(\mathbf{v}) d\mathbf{v}$$

Energy deposition :

$$-N_e \int \mathbf{v} \left(O(\mathbf{v}-\mathbf{U}).\frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}) + \mathbf{r}(\mathbf{v}-\mathbf{U}) f(\mathbf{v}) \right) d\mathbf{v} = \mathbf{R}.\mathbf{U} - N_e \int \mathbf{w} \left(O(\mathbf{w}).\frac{\partial}{\partial \mathbf{w}} f(\mathbf{w}) + \mathbf{r}(\mathbf{w}) f(\mathbf{w}) \right) d\mathbf{w}$$

Estimation of $\mathbf{R}$ in a zone M

$$\int_M \mathbf{R}(x) dx \delta t \simeq \sum_{p, X_p \cap M \neq 0} w_p (V_{p+\delta t} - V_p), \text{ without Brownian motion}$$

Estimation of energy

$$\int_M dx \int \mathbf{w} \left(O(\mathbf{w}).\frac{\partial}{\partial \mathbf{w}} f(\mathbf{w}) + \mathbf{r}(\mathbf{w}) f(\mathbf{w}) \right) d\mathbf{w} \delta t \simeq \sum_{p, X_p \cap M \neq 0} w_p (|W_{p+\delta t}|^2 - |W_p|^2)$$