gPC FEM for stochastic elliptic PDEs

C. Schwab
R.A. Todor, P. Frauenfelder
Seminar for Applied Mathematics
ETH Zürich
CEMRACS 2006, CIRM, Luminy, France

Outline

1 Random Fields, Statistics

2 Stochastic BVP

3 Karhunen-Loève (KL) Expansion

4 Reduction to High Dimensional Deterministic BVP

5 Numerical Experiments

Literature

Perturbation Methods

J. B. Keller (1964)

M. Kleiber, T.D. Hien (1992)

Stochastic Galerkin; Generalized Polynomial Chaos (Karhunen-Loève)

N. Wiener (1930ies), Cameron and Martin (1950)

R. G. Ghanem, P. D. Spanos (1991 –)

J.T. Oden, I. Babuska et al. (2001 -)

G. E. Karniadakis et al. ($\sim$ 2000 -)

Babuška, Tempone and Zouraris (SINUM 2004)

H. Matthies and R. Keese (Review; CMAME 2004)

C. Schwab, P. Frauenfelder and R.A. Todor (CMAME 2005)

R.A. Todor (SINUM 2004 and Diss. ETHZ 2005)

C. Schwab and R.A. Todor (JCP 2006)

Random Fields, Statistics

$D \subset \mathbb{R}^d$ bounded Lipschitz domain, (Ω, Σ, P) probability space

Random field on D

$$u(x,\omega) : D \times \Omega \to \mathbb{R} \quad \text{jointly measurable w.r.t. } \lambda \times P$$

Statistics

Mean Field $\begin{cases} E_u : D \to \mathbb{R} \\ E_u(x) := \int_\Omega u(x,\omega)\, dP(\omega) \end{cases}$

Covariance $\begin{cases} C_u : D \times D \to \mathbb{R} \\ C_u(x,x') := \int_\Omega (u(x,\omega) - E_u(x))(u(x',\omega) - E_u(x'))\, dP(\omega) \end{cases}$

Probabilistic Level Sets $\begin{cases} \text{for } \varepsilon > 0, u_0 \in \mathbb{R} \\ D^\varepsilon_{u_0} := \{x \in D \mid P(u(x,\cdot) > u_0) > \varepsilon\} \end{cases}$

Stochastic BVP

Given: random field a on D, deterministic source term $f : D \to \mathbb{R}$

(SBVP) Find random field $u(x,\omega)$ satisfying

$$\left\{ \begin{array}{rcll} -\mathrm{div}(a(x,\omega)\nabla_x u(x,\omega)) & = & f(x) & \text{in } D \\ u(x,\omega) & = & 0 & \text{on } \partial D \end{array} \right. \quad P- \text{ a.e. } \omega \in \Omega$$

If $a(x,\omega)$ is bounded, positive on $D \times \Omega$ $\lambda \otimes P - a.s.$ and if $f \in H^{-1}(D)$, then (SBVP) is well-posed in $H^1(D) \otimes L^2(\Omega)$

Statistical version

Given: statistics of a $(E_a, C_a, \ldots)$ and of f

Find: statistics of u $(E_u, C_u, D^{\varepsilon}_{u_0}, \ldots)$

Karhunen-Loève Expansion

$H_1, \langle , \rangle_{H_1}$, $H_2, \langle , \rangle_{H_2}$ and $S, \langle , \rangle_S$ separable Hilbert spaces,

$(s_m)_{m\in\Lambda}$ ONB in S.

Then any $f \in H_1 \otimes S$ can be written as

$$f = \sum_{m\in\Lambda} f_m \otimes s_m.$$

$$H_1 \otimes S \times H_2 \otimes S \ni (f,g) \quad \longrightarrow \quad C_{fg} := \sum_{m\in\Lambda} f_m \otimes g_m \in H_1 \otimes H_2$$

well-defined, bilinear and bounded with norm 1, and ind. of the choice of the basis $(s_m)_{m\in\Lambda}$ in S.

Karhunen-Loève Expansion

Definition (Correlation)
For $f \in H_1 \otimes S$ and $g \in H_2 \otimes S$, $C_{fg} \in H_1 \otimes H_2$ is the **correlation** of the pair (f, g).

Remark
If $H_1 = H_2 = H$ and the corresponding scalar products also coincide, the set

$$\{C_f := C_{ff} \mid f \in H \otimes S\}$$

of all correlation kernels is in one-to-one correspondence with a certain class of operators on H.

Karhunen-Loève Expansion

Proposition 1 (Karhunen-Loève)
$H, \langle , \rangle_H$ and $S, \langle , \rangle_S$ separable Hilbert spaces,

$$\mathrm{dim} H = \mathrm{dim} S$$

$(s_m)_{m\in\Lambda}$ ONB in S.

Then correlations of elements in $H \otimes S$ are in a one-to-one correspondence with the positive definite trace class operators in H, via

$$\sum_{m\in\Lambda} f_m \otimes f_m = C_f \longrightarrow \mathcal{C}_f : H \ni x \to \sum_{m\in\Lambda} \langle f_m, x\rangle_H \cdot f_m \quad \in H, \quad (1)$$

for $f = \sum_{m\in\Lambda} f_m \otimes s_m$.

Proof.

(i) $\mathcal{C}_f$ defined on the r.h.s. of (1) is compact (norm limit of finite rank operators obtained by truncating the series).

(ii) Positivity of $\mathcal{C}_f$ clear

(iii) Trace is finite: choosing $(e_m)_{m\in\Lambda}$ ONB in H,

$$\text{Tr } \mathcal{C}_f = \sum_{m\in\Lambda} \langle \mathcal{C}_f e_m, e_m \rangle_H = \sum_{m\in\Lambda} \sum_{n\in\Lambda} \langle f_m, e_n \rangle_H^2 = \sum_{m\in\Lambda} \|f_m\|_H^2 = \|f\|_{H\otimes S}^2 < \infty.$$

The mapping (1) is therefore well-defined.

(iv) From

$$\langle \mathcal{C}_f x, y \rangle_H = \langle C_f, x \otimes y \rangle_{H\otimes H} \quad \forall x, y \in H \tag{2}$$

it follows that the definition of $\mathcal{C}_f$ does not depend on the basis $(s_m)_{m\in\Lambda}$ and that the mapping (1) is injective.

(v) Surjectivity of (1):
Let $\mathcal{C}$ be a positive definite trace class operator in H. Then $\mathcal{C}$ is compact and has eigenpair sequence $(\lambda_m, \phi_m)_{m\in\Lambda}$,

$$\mathcal{C}\phi_m = \lambda_m \phi_m \quad \forall m \in \Lambda. \tag{3}$$

The eigenvalues $(\lambda_m)_{m\in\Lambda}$ have finite multiplicity, sequence is nonincreasing and may accumulate only in 0.

Moreover, the trace class condition reads

$$\sum_{m\in\Lambda} \lambda_m < \infty. \tag{4}$$

Then the series

$$\sum_{m\in\Lambda} \sqrt{\lambda_m} \cdot \phi_m \otimes s_m \tag{5}$$

converges due to (4) to an element $f \in H \otimes S$ for which

$$C_f = \sum_{m\in\Lambda} \lambda_m \cdot \phi_m \otimes \phi_m \tag{6}$$

From (2),(3), (6) it follows that $\mathcal{C}_f$ has the same spectral decomposition as $\mathcal{C}$, i.e. $\mathcal{C}_f = \mathcal{C}$. ■

Karhunen-Loève Expansion

Corollary
Let $H, \langle , \rangle_H$ be a separable Hilbert space and $C \in H \otimes H$ be a correlation kernel.

Then in terms of the spectral decomposition of $\mathcal{C} \in \mathcal{B}_\infty(\mathcal{H})$ defined as in (2), C can be represented as

$$C = \sum_{m \in \Lambda} \lambda_m \cdot \phi_m \otimes \phi_m. \tag{7}$$

Karhunen-Loève Expansion

Theorem (Karhunen - Loève)

$H, \langle , \rangle_H$, $S, \langle , \rangle_S$ separable Hilbert spaces

$C \in H \otimes H$ correlation kernel, representation (7).

Then $f \in H \otimes S$ satisfies $C_f = C$ iff there exists an orthonormal family $(X_m)_{m\in\Lambda} \subset S$, such that

$$f = \sum_{m\in\Lambda} \sqrt{\lambda_m}\, \phi_m \otimes X_m. \tag{8}$$

Proof.

The 'if' part follows by the arguments used to conclude the proof of the Theorem, after completing the family $(X_m)_{m\in\Lambda}$ to an ONB.

Conversely, if $C_f = C$, then we expand

$$f = \sum_{m\in\Lambda} \phi_m \otimes Y_m, \tag{9}$$

with $(Y_m)_{m\in\Lambda} \subset S$, from which it follows via Proposition 1

$$C_f = \sum_{m,m'\in\Lambda} \langle Y_m, Y'_m \rangle_S \cdot \phi_m \otimes \phi'_m \tag{10}$$

Comparing (10) and (7), it follows that

$$\langle Y_m, Y_{m'} \rangle_S = \lambda_m \delta_{mm'} \tag{11}$$

and (8) holds with $X_m := Y_m/\sqrt{\lambda_m}$.

Karhunen-Loève expansion

- separation of deterministic and stochastic variables -

Proposition 1 (Karhunen-Loève)
If $a \in L^2(D \times \Omega)$ then in $L^2(D \times \Omega)$

$$a(x,\omega) = E_a(x) + \sum_{m \geq 1} \sqrt{\lambda_m}\phi_m(x) X_m(\omega)$$

where
$(\lambda_m, \phi_m)_{m \geq 1}$ the eigenpair sequence of C_a compact, selfadjoint

$$C_a : L^2(D) \to L^2(D) \qquad (C_a v)(x) := \int_D C_a(x,x')v(x')dx' \quad \forall v \in L^2(D)$$

$(X_m)_{m \geq 1}$ vanishing mean, uncorrelated rv's

$$\int_\Omega X_m(\omega)\, dP(\omega) = 0 \qquad \int_\Omega X_n(\omega) X_m(\omega)\, dP(\omega) = \delta_{nm}, \quad \forall n, m \geq 1$$

Karhunen-Loève expansion

- convergence -

$$a(x,\omega) = E_a(x) + \sum_{m\geq 1} \sqrt{\lambda_m}\phi_m(x)X_m(\omega)$$

KL expansion converges in $L^2(D \times \Omega)$, not necessarily in $L^\infty(D \times \Omega)$

To ensure $L^\infty(D \times \Omega)$ convergence, must

- estimate λ_m

- estimate $||\phi_m||_{L^\infty(D)}$

- assume bounds for $||X_m||_{L^\infty(\Omega)}$

Karhunen-Loève expansion

- eigenvalue estimates -

Regularity of C_a ensures decay of KL-eigenvalue sequence $(\lambda_m)_{m\geq 1}$

Definition
A correlation function $V_a : D\times D \to \mathbb{R}$ is **piecewise analytic/H^k/ C^k on** $D\times D$ if there exists a partition $\mathcal{D} = \{D_j\}_{j=1}^J$ of D into a finite sequence of simplices D_j such that

$$\overline{D} = \bigcup_{j=1}^{J} \overline{D_j} \tag{12}$$

and such that V is analytic in an open neighbourhood of $\overline{D_j} \times \overline{D_{j'}}$ / is in $H^k(D_j, L^2(D_{j'}))$/ is in $C^k(\overline{D}_j, L^2(D_{j'}))$ for any pair (j, j').

Karhunen-Loève expansion

- eigenvalue estimates -

Lemma
Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathcal{C} \in \mathcal{B}(\mathcal{H})$ be symmetric, nonnegative and compact operator whose eigenpair sequence is denoted by $(\lambda_m, \phi_m)_{m\geq 1}$.
If $m \in \mathbb{N}$ and $\mathcal{C}_m \in \mathcal{B}(\mathcal{H})$ is an operator of rank at most m, then it holds

$$\lambda_{m+1} \leq \|\mathcal{C} - \mathcal{C}_m\|_{\mathcal{B}(\mathcal{H})}. \tag{13}$$

Proof. Straightforward application of the minimax principle:

$$\lambda_{m+1} = \min_{\substack{V\subset H \\ \dim V^\perp \le m}} \max_{\substack{\phi\in V \\ \|\phi\|_H=1}} \langle \mathcal{C}\phi, \phi\rangle \quad \le \quad \max_{\substack{\phi\in(\mathcal{C}_m)^\perp \\ \|\phi\|_H=1}} \langle \mathcal{C}\phi, \phi\rangle$$

$$= \quad \max_{\substack{\phi\in(\mathcal{C}_m)^\perp \\ \|\phi\|_H=1}} \langle (\mathcal{C} - \mathcal{C}_m)\phi, \phi\rangle \le \|\mathcal{C} - \mathcal{C}_m\|_{\mathcal{B}(\mathcal{H})}$$

■

Karhunen-Loève expansion

- eigenvalue estimates -

Theorem 2
Exponential decay (e.g. Gaussian covariance kernel $C_a(x,x') := \sigma^2 \exp(-\gamma|x-x'|^2)$)

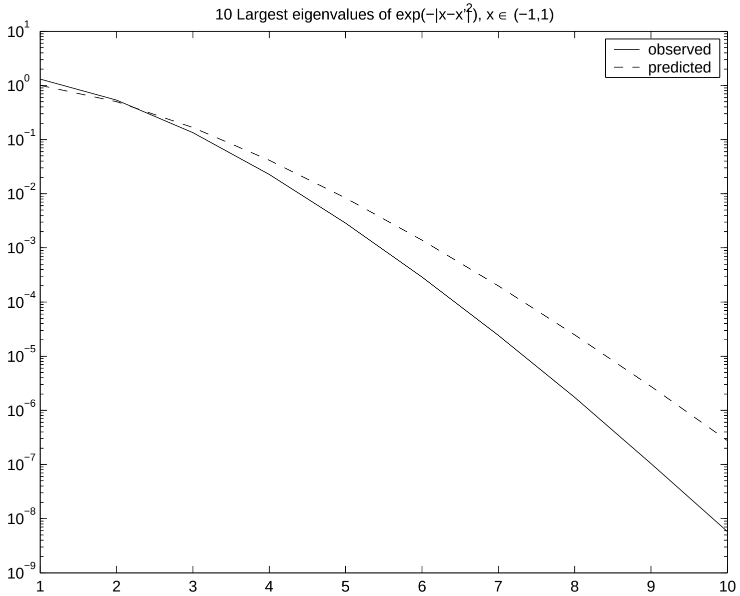
C_a pw analytic on $D \times D \Longrightarrow \exists c > 0 \quad 0 \leq \lambda_m \lesssim \exp(-cm^{1/d}) \quad \forall m \geq 1$

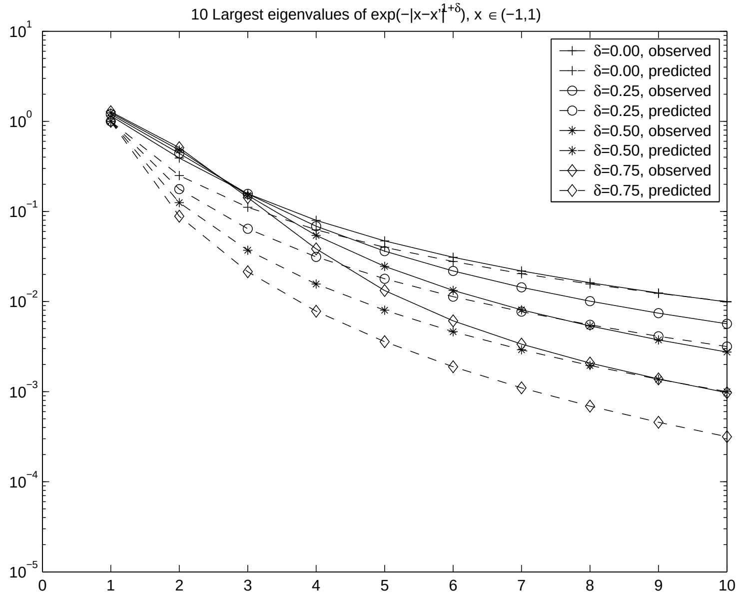
Algebraic decay (e.g. exponential covariance kernel $C_a(x,x') := \sigma^2 e^{-\gamma|x-x'|}$)

C_a pw $H^p(D) \otimes L^2(D)$ $(p \geq 1) \Longrightarrow \quad 0 \leq \lambda_m \lesssim m^{-(2p-1)/d} \quad \forall m \geq 1$

Karhunen-Loève expansion

- eigenvalue estimates -





Karhunen-Loève expansion

- eigenfunction estimates -

Regularity of C_a ensures L^∞ bounds for L^2-scaled eigenfunctions $(\phi_m)_{m\geq 1}$

Theorem 3

C_a pw smooth on $D\times D \Longrightarrow \forall s>0, \quad \|\phi_m\|_{L^\infty(D)} \lesssim |\lambda_m|^{-s} \quad \forall m \geq 1$

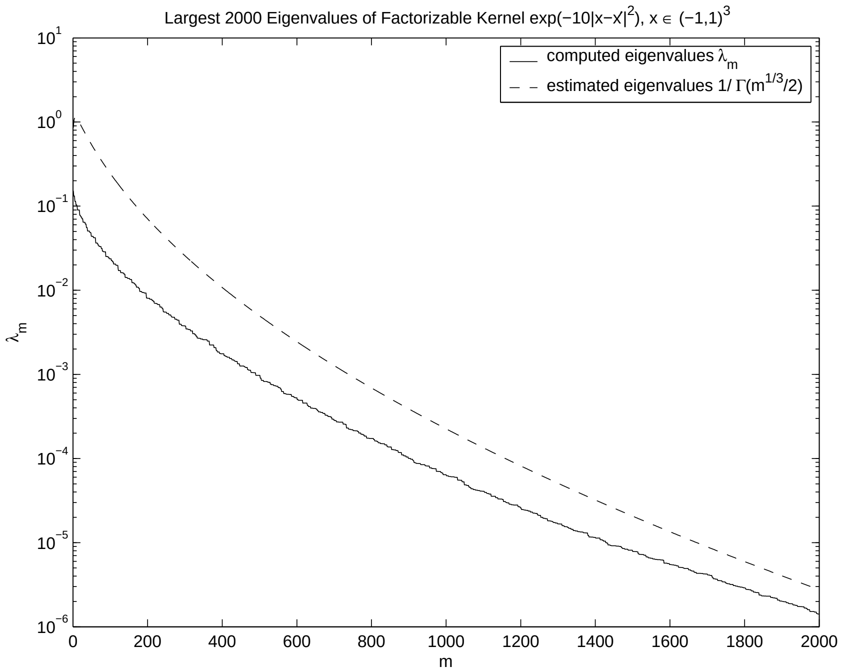
Karhunen-Loève expansion

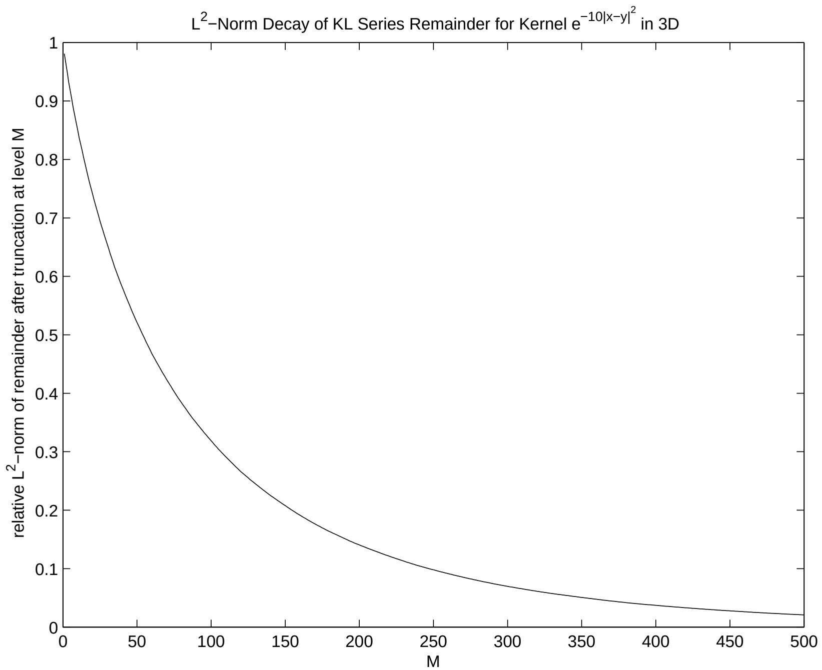
- convergence rate -

Conclusion: KL expansion converges uniformly and exponentially on $D \times \Omega$ if
- C_a pw analytic
- $(X_m)_{m \geq 1}$ uniformly bounded on Ω (e.g. uniformly distributed in $(-1/2, 1/2)$)

Karhunen-Loève expansion

- convergence rate -





Karhunen-Loève expansion

- computation of eigenpairs -

Find $(\lambda, \phi) \in \mathbb{R} \times L^2(D)$ such that

$$\int_D \psi(x) \int_D C_a(x, x')\phi(x')\, dx' dx = \lambda \int_D \psi(x)\phi(x) dx, \quad \forall \psi \in L^2(D)$$

Discretization using $S^{p,-1}(D, \mathcal{T}_h) \subset L^2(D)$

Theorem 4
If C_a pw smooth and $\mathcal{T}$ subordinate to smoothness partition of $a(x, \omega)$, then $\forall m \geq 1 \quad \exists C_m > 0$ such that

$$\forall h > 0 \qquad \|\phi_m - \phi_{m,h}\|_{L^\infty(D)} \leq C_m h^{p+1}, \quad |\lambda_m - \lambda_{m,h}| \leq C_m h^{p+1}$$

Open Problem: $\forall m \geq 1: \quad C_m \leq \Phi(m)$, $\Phi =?$

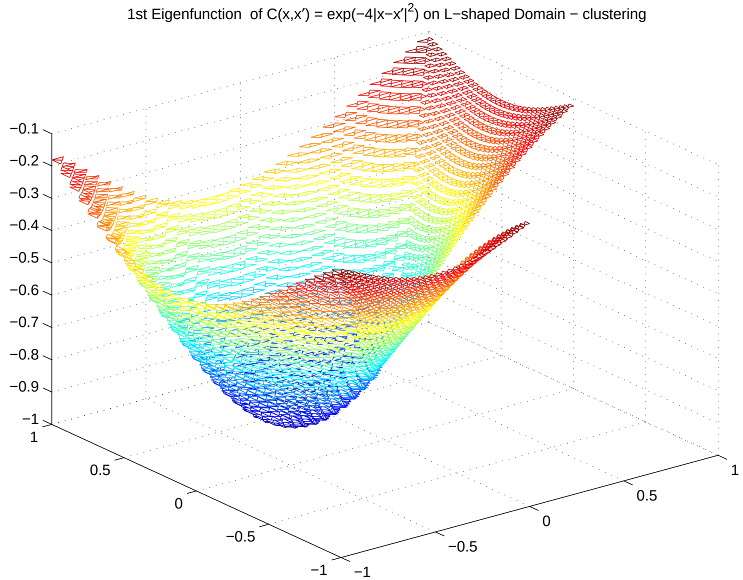
Kernel independent multipole method, Čhebysev-clustering ($D \times D =$ near field $\cup$ far field) $\Rightarrow$ stiffness matrix (approximated) : sparse + low rank matrices

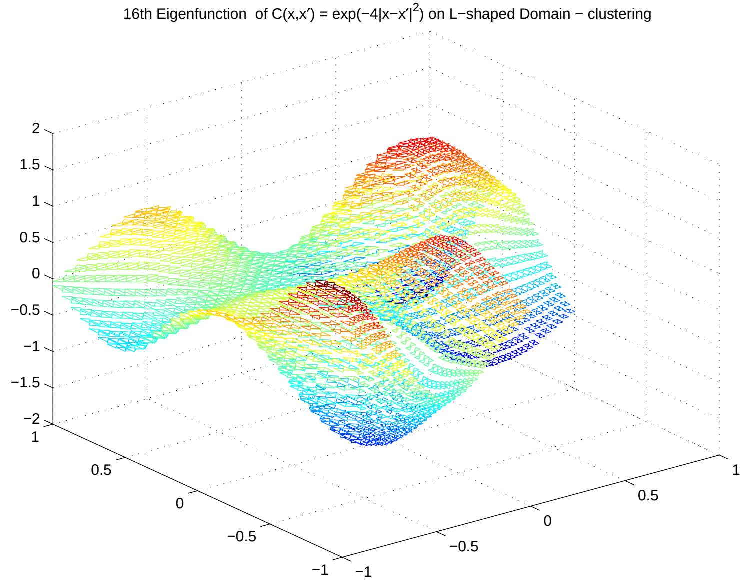
Work(Matrix * Vector) $= O(N \log^c N)$,

Computation of M eigenfunctions: work is $O(MN \log^{c'} N)$.

Karhunen-Loève expansion

- computation of eigenpairs -





Karhunen-Loève expansion

- truncation -

$M \in \mathbb{N}$ KL-truncation order

$$a_M(x,\omega) = E_a(x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\phi_m(x) X_m(\omega)$$

$(SBVP)$ with stochastic coefficient $a(x,\omega)$

$$-\mathrm{div}(a(x,\omega)\nabla_x u(x,\omega)) = f(x) \quad \text{in } H^{-1}(D) \otimes L^2(\Omega)$$

$(SBVP)_M$ with truncated stochastic coefficient $a_M(x,\omega)$

$$-\mathrm{div}(a_M(x,\omega)\nabla_x u_M(x,\omega)) = f(x) \quad \text{in } H^{-1}(D) \otimes L^2(\Omega)$$

Theorem 5 (Sc. and Todor 2003)
If C_a pw analytic and $(X_m)_{m\geq 1}$ uniformly bounded, then ex. $M_0 > 0$ such that $(SBVP)_M$ well-posed for $M \geq M_0$ and

$$\|u - u_M\|_{H_0^1(D)\otimes L^2(\Omega)} \lesssim \exp(-cM^{1/d}) \quad \forall M \geq M_0$$

Reduction to high dimensional deterministic bvp

$$a_M : D \times \Omega \to \mathbb{R}, \quad a_M(x,\omega) = E_a(x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\phi_m(x)X_m(\omega)$$

Assumption
$(X_m)_{m\geq 1}$ independent, uniformly bounded family of rv's
(e.g. X_m uniformly distributed in $I = (-1/2, 1/2), \forall m$)

$$\begin{array}{rcl} \text{Random variable } X_m & \longrightarrow & \text{Parameter } y_m \in I \\ (X_1, X_2, \ldots, X_M) & \longrightarrow & y = (y_1, y_2, \ldots, y_M) \in I^M \end{array}$$

$$\tilde{a}_M : D \times I^M \to \mathbb{R}, \quad \tilde{a}_M(x,y) = E_a(x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\phi_m(x)y_m$$

Reduction to high dimensional deterministic bvp

stochastic bvp

$$-\text{div}(a_M(x,\omega)\nabla_x u_M(x,\omega)) = f(x) \quad \text{in } H^{-1}(D), \quad P-\text{a.e. } \omega \in \Omega$$

parametric deterministic bvp

$$-\text{div}(\tilde{a}_M(x,y)\nabla_x \tilde{u}_M(x,y)) = f(x) \quad \text{in } H^{-1}(D), \quad \forall y \in I^M$$

Theorem 6 (Babuška, Tempone, Zouraris (SINUM 2004))
Under **Assumption**, the parametric deterministic bvp is well-posed and

$$u_M(x,\omega) = \tilde{u}_M(x, X_1(\omega), X_2(\omega), \ldots, X_M(\omega))$$

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization -

$$\tilde{a}_M(x,y) = E_a(x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\phi_m(x) y_m$$

parametric deterministic bvp

$$-\mathrm{div}(\tilde{a}_M(x,y)\nabla_x \tilde{u}_M(x,y)) = f(x) \quad \text{ in } H^{-1}(D)\otimes L^2(I^M)$$

Galerkin semi-discretization in y ("Least-squares innerproduct")

$\tilde{a}_M(x,y)$ affine in y $\Rightarrow$ $\tilde{u}_M(x,y)$ analytic in y $\Rightarrow$ p-FEM in y

task: solve dbvp with an accuracy* $O(e^{-cM^{1/d}})$ in "low complexity"**
*how to choose the polynomial space $\mathcal{P}$ in $y = (y_1, y_2, \ldots, y_M)$?
**how to choose a basis $\mathcal{B}$ of $\mathcal{P}$?

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization -

$\mathcal{P} \subset L^2(I^M)$ polynomial space, $\mathcal{B}$ basis of $\mathcal{P}$

Theorem 7 (BTZ , Sc. and Todor 2002)
If $\mathcal{P}$ of tensor product type, then
i. ($\mathcal{P}$): Accuracy $O(\exp(-cM^{1/d}))$ can be obtained by choosing
- uniform polynomial degree

$$p = M \quad \text{in} \quad y_1, \ldots, y_M \Rightarrow \dim \mathcal{P} \sim (M+1)^M \quad \text{dBVPs}$$

or
- variable polynomial degree

$$p_m = \lceil M/m \rceil \quad \text{in} \quad y_m \Rightarrow \dim \mathcal{P} \sim e^{O(M)}$$

ii. ($\mathcal{B}$): $\exists \mathcal{B}$ of tensor product type: allows decoupling the problem in y

solving the dbvp in $D \times I^M \quad \Longleftrightarrow \quad$ parallel solution of Card $\mathcal{B}$ dBVPs in D

$\mathcal{P}$ not of tensor product type (sparse): $\mathcal{P}$ =?, $\mathcal{B}$ =?

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

$I = (-1/2, 1/2)$, $p \geq 0$ fixed
$V_l :=$ pw polynomials of degree $p \geq 0$ on a mesh of width 2^{-l} in I

Theorem 8 (Todor and Sc. 2005)
If C_a pw analytic, then there exists a *sparse* tensor product space

$$\widehat{V}_M \subset \underbrace{V_M \otimes V_M \otimes \cdots \otimes V_M}_{M \text{ times}}$$

such that
i. ($\mathcal{P}$): The following approximation property holds,

$$\begin{aligned}
\inf_{v \in \widehat{V}_M} \|\tilde{u}_M - v\|_{H_0^1(D)\otimes L^2(I^M)} &\lesssim \exp(-cM^{1/d} + o(M^{1/d})) \\
\widehat{N} := \dim \widehat{V}_M &\lesssim \exp(cM^{1/d}/(p+1) + o(M^{1/d})) \\
\inf_{v \in \widehat{V}_M} \|\tilde{u}_M - v\|_{H_0^1(D)\otimes L^2(I^M)} &\lesssim \widehat{N}^{-p-1}
\end{aligned}$$

ii. ($\mathcal{B}$): Products of 1-d discontinuous Multiwavelets of degree p

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

Stochastic Regularity: Piecewise analyticity of correlation and bounded range of X_m ensure

$$0 \le \rho_m := \|\psi_m\|_{L^\infty(D)} \le c_r \exp(-c_{1,r} m^\kappa) \quad \forall m \in \mathbb{N}_+. \tag{14}$$

Proposition
If $\tilde{u}_M$ solves $(SBVP)_M$,

$$\|\partial_y^\alpha \tilde{u}_M(y,\cdot)\|_{H_0^1(D)} \le a_-^{-|\alpha|} |\alpha|! \prod_{m=1}^{M} \rho_m^{\alpha_m} \cdot \|\tilde{u}_M(y,\cdot)\|_{H_0^1(D)}, \tag{15}$$

$\forall y \in I^M, \forall \alpha \in \mathbb{N}^M, M \in \mathbb{N}.$

Proof. Induction on $|\alpha|$.

Since (15) is clear for $|\alpha| = 0$, assume it for all $\alpha \in \mathbb{N}^M$ such that $|\alpha| \leq k$, for some $k \in \mathbb{N}$.

Consider a multiindex α such that $|\alpha| = k + 1$ and we apply ∂_y^α to $(SBVP)_M$.

We obtain

$$-\operatorname{div}(\tilde{a}_M(x,y)\nabla\partial_y^\alpha\tilde{u}_M(x,y)) = \sum_{m=1}^{M} \alpha_m \operatorname{div}(\sqrt{\lambda_m}\phi_m(x)\nabla\partial_y^{\alpha-e_m}\tilde{u}_M(x,y)) \tag{16}$$

from which it follows

$$a_-\|\partial_y^\alpha\tilde{u}_M(\cdot,y)\|_{H_0^1(D)} \leq \sum_{m=1}^{M} \alpha_m\rho_m\|\partial_y^{\alpha-e_m}\tilde{u}_M(\cdot,y)\|_{H_0^1(D)} \tag{17}$$

The desired estimate follows then by using (15) in (17) for all multiindices $\alpha - e_m$, $1 \leq m \leq M$, whose length equals k. ∎

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

Notations.
For $p \in \mathbb{N}_+$, $l \in \mathbb{N}$, denote by $V^{l,p}$ the space of pw polynomials of degree at most $p-1$ on a regular mesh of width 2^{-l} in I.

Further set $V^{-1,p} := \{0\}$ and define by

$$W^{l,p} := V^{l,p} \ominus V^{l-1,p} \tag{18}$$

hierarchical excess of the scale $(V^{l,p})_{l\in\mathbb{N}}$, in the sense of $L^2(I)$.

$L^2(I)$ orthogonal decomposition:

$$L^2(I) = \bigoplus_{l=0}^{\infty} W^{l,p}. \tag{19}$$

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

P_V : $L^2(I)$ projection onto closed subspace V of $L^2(I)$, then $(V^{l,p})_{l\in\mathbb{N}}$

$$\|u - P_{V^{l,p}}u\|_{L^2(I)} \leq c_p 2^{-lp} \|\partial^p u\|_{L^2(I)} \quad \forall u \in H^p(I), \tag{20}$$

with some constant $c_p > 0$.

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

Build the FE spaces in I^M as tensor products:
For $\mathbf{l} = (l_1, l_2, \ldots, l_M) \in \mathbb{N}^M$ introduce

$$W^{\mathbf{l},p} := \bigotimes_{m=1}^{M} W^{l_m,p}, \tag{21}$$

which enables us via (19) to decompose $L^2(I^M)$ as

$$L^2(I^M) = \bigoplus_{\mathbf{l}\in\mathbb{N}^M} W^{\mathbf{l},p}. \tag{22}$$

Equivalently,

$$u = \sum_{\mathbf{l}\in\mathbb{N}^M} u^{\mathbf{l}}, \quad u^{\mathbf{l}} := P_{W^{\mathbf{l},p}} u \quad \forall u \in L^2(I^M). \tag{23}$$

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

For a multiindex $\mathbf{l} = (l_1, l_2, \ldots, l_M) \in \mathbb{N}^M$, we define its length $|\mathbf{l}|$ by

$$|\mathbf{l}| := l_1 + l_2 + \cdots + l_M. \tag{24}$$

Further,

$$\mathcal{J}_{\mathbf{l}} := \{m \mid 1 \leq m \leq M,\, l_m > 0\}, \quad j_{\mathbf{l}} := |\mathcal{J}_{\mathbf{l}}|, \tag{25}$$

so that $\mathcal{J}_{\mathbf{l}} = \{m_1, m_2, \ldots, m_{j_{\mathbf{l}}}\}$.

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

Proposition (Component Size Estimate)
If $\tilde{u}_M$ solves $(SBVP)_M$ and X_m bounded, then

$$\|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} \leq c_{a,p}^{j_1} \cdot 2^{-|\mathbf{l}|p} \cdot (pj_1)! \cdot \prod_{j=1}^{j_1} \rho_{m_j}^p \cdot \|\tilde{u}_M\|_{L^2(I^M)}, \tag{26}$$

where $\tilde{u}_M^{\mathbf{l}} := P_{W^{\mathbf{l},p}} \tilde{u}_M \, \forall \mathbf{l} \in \mathbb{N}^M$.

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

Proof: For a fixed multiindex $\mathbf{l} \in \mathbb{N}^M$ we define $\mathbf{e} := (e_1, e_2, \ldots, e_M) \in \mathbb{N}^M$ (depending on $\mathbf{l}$) by

$$e_m := \begin{cases} 1 & if \quad l_m > 0 \\ 0 & if \quad l_m = 0 \end{cases} \qquad \forall 1 \leq m \leq M. \tag{27}$$

We write

$$\begin{aligned} \tilde{u}_M^{\mathbf{l}} = P_{W^{\mathbf{l},p}} \tilde{u}_M &= \bigotimes_{m=1}^{M} (P_{V^{l_m,p}} - P_{V^{l_m-1,p}}) \tilde{u}_M \\ &= \bigotimes_{m=1}^{M} (P_{V^{l_m,p}} - I + I - P_{V^{l_m-1,p}}) \tilde{u}_M \\ &= \sum_{\mathbf{f} \in \mathbb{N}^M, \mathbf{f} \leq \mathbf{e}} (-1)^{M-|\mathbf{f}|} \bigotimes_{m=1}^{M} (I - P_{V^{l_m - f_m,p}}) \tilde{u}_M. \end{aligned} \tag{28}$$

Using the approximation property (20) and noting that the sum in (28) consists of 2^{j_1} terms, we can estimate

$$\begin{aligned}
\|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} &\le \sum_{\mathbf{f}\in\mathbb{N}^M,\,\mathbf{f}\le\mathbf{e}} c_p^{j_1} 2^{-(|\mathbf{l}|-|\mathbf{f}|)p} \cdot \|\partial_y^{p\cdot\mathbf{e}}\tilde{u}_M\|_{L^2(I^M)} \\
&\le \sum_{\mathbf{f}\in\mathbb{N}^M,\,\mathbf{f}\le\mathbf{e}} (2^p c_p)^{j_1} 2^{-|\mathbf{l}|p} \cdot \|\partial_y^{p\cdot\mathbf{e}}\tilde{u}_M\|_{L^2(I^M)} \\
&\le (2^{p+1} c_p)^{j_1} 2^{-|\mathbf{l}|p} \cdot \|\partial_y^{p\cdot\mathbf{e}}\tilde{u}_M\|_{L^2(I^M)}. \qquad (29)
\end{aligned}$$

Using the regularity estimate (15) in (29) leads to the desired estimate (26).
■

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

Definition of 'supersparse' Tensor Product Spaces:
For $\mu, \nu \in \mathbb{N}$ we introduce the index set

$$\boldsymbol{\Sigma}_{\mu,\nu} \subset \mathbb{N}^M,\ \boldsymbol{\Sigma}_{\mu,\nu} := \{\mathbf{l} \in \mathbb{N}^M \mid |\mathbf{l}| \leq \mu, \mathbf{l} \text{ has at most } \nu \text{ nontrivial entries}\}, \quad (30)$$

and define 'supersparse' tensor subspace of $L^2(I^M)$,

$$\widehat{V}^{\mu,\nu} := \bigoplus_{\mathbf{l} \in \boldsymbol{\Sigma}_{\mu,\nu}} W^{\mathbf{l},p}. \quad (31)$$

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: h FEM -

Theorem 8 (Restatement)
There exist positive constants c_3, c_4 and c_r such that for

$$\mu = \lceil c_4 M^\kappa \rceil, \quad \nu = \lceil c_3 M^{\kappa/(\kappa+1)} \rceil \tag{32}$$

it holds

$$\|\tilde{u}_M - P_{\widehat{V}^{\mu,\nu}} \tilde{u}_M\|_{L^2(I^M)} \leq c_{a,r,p} \exp(-c_r M^\kappa + o(M^\kappa)) \tag{33}$$

and

$$\dim \widehat{V}^{\mu,\nu} \leq c_{a,r,p} \exp(\frac{c_r}{p} M^\kappa + o(M^\kappa)). \tag{34}$$

Note the same constant c_r appears in (33) and (34) respectively.

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: p FEM -

Proof. For $c_3, c_4 > 0$ which will be chosen later and with μ, ν as in (32), we write

$$
\begin{aligned}
\|\tilde{u}_M - P_{\widehat{V}^{\mu,\nu}} \tilde{u}_M\|_{L^2(I^M)} &\leq \sum_{\mathbf{l} \in \mathbb{N}^M \setminus \Lambda^{\mu,\nu}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} \\
&= \sum_{\substack{\mathbf{l} \in \mathbb{N}^M \\ j_{\mathbf{l}} > \nu}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} + \sum_{\substack{\mathbf{l} \in \mathbb{N}^M \\ j_{\mathbf{l}} \leq \nu \\ |\mathbf{l}| > \mu}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} \qquad (35)
\end{aligned}
$$

We estimate each of the two sums in (35) separately.

In both cases we use component size estimate (26) and notations (24), (25).

$$
\begin{aligned}
\sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ j_{\mathbf{l}}>\nu}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} &= \sum_{j=\nu+1}^{M} \sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ |j_{\mathbf{l}}|=j}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} \\
&\leq \sum_{j=\nu+1}^{M} (2^{p+1} c_p / a_-^p)^j \cdot (pj)! \cdot \sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ |j_{\mathbf{l}}|=j}} 2^{-|\mathbf{l}|p} \cdot \prod_{k=1}^{j} \rho_{m_k}^p \cdot \|\tilde{u}_M\|_{L^2(I^M)} \\
&\leq \sum_{j=\nu+1}^{M} (2^{p+1} c_p / a_-^p)^j \cdot (pj)! \cdot \sum_{1\leq m_1<\ldots<m_j\leq M} \prod_{k=1}^{j} e^{-c_r m_k^\kappa p} \cdot \\
&\quad \cdot \sum_{l_{m_1},\ldots,l_{m_j}=1}^{\infty} 2^{-p(l_{m_1}+\cdots+l_{m_j})} \cdot \|\tilde{u}_M\|_{L^2(I^M)} \\
&\leq \sum_{j=\nu+1}^{M} x^j (pj)! \cdot \sum_{1\leq m_1<\ldots<m_j\leq M} \prod_{k=1}^{j} e^{-c_r m_k^\kappa p} \cdot \|\tilde{u}_M\|_{L^2(I^M)} \qquad (36)
\end{aligned}
$$

with some $x > 0$ depending on p, a_-.

We then use

Lemma A If $\kappa > 0$, and $x > y > z > 0$, then there exist $c_{\kappa,x,y}, c_{\kappa,y,z} > 0$ such that

$$c_{\kappa,x,y} \exp(-x\frac{1}{1+\kappa}j^{1+\kappa}) \leq \sum_{1\leq m_1<\ldots<m_j<\infty} \prod_{k=1}^{j} \exp(-ym_k^\kappa) \leq c_{\kappa,y,z} \exp(-z\frac{1}{1+\kappa}j^{1+\kappa}) \tag{37}$$

$\forall j \in \mathbb{N}_+$.

Use Lemma A in (36) to obtain

$$\sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ j_\mathbf{l}>\nu}} \|\tilde{u}_M^\mathbf{l}\|_{L^2(I^M)} \lesssim \sum_{j=\nu+1}^{M} a^{(1+\kappa)j^{1+\kappa}p} \cdot \|\tilde{u}_M\|_{L^2(I^M)}, \tag{38}$$

for any $a \in (e^{-c_r}, 1)$, and with a constant depending on a, r, p.

The series in (38) converges faster than geometrically, therefore we conclude, with ν as in (32) and c_3 to be chosen next,

$$\sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ j_\mathbf{l}>\nu}} \|\tilde{u}_M^\mathbf{l}\|_{L^2(I^M)} \lesssim a^{(1+\kappa)c_3^{1+\kappa}M^\kappa p} \cdot \|\tilde{u}_M\|_{L^2(I^M)}, \tag{39}$$

with a constant depending on a, r, p.

We choose now c_3 to match the r.h.s. of (33), i.e. such that

$$a^{(1+\kappa)c_3^{1+\kappa}p} = e^{-c_r}. \tag{40}$$

We turn to the second sum in (35).

We use again bound (26) and Lemma A.

We write

$$\begin{aligned}
\sum_{\substack{\mathbf{l}\in\mathbb{N}^M\\ j_{\mathbf{l}}\le\nu\\ |\mathbf{l}|>\mu}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} &\le \sum_{\substack{\mathbf{l}\in\mathbb{N}^M\\ j_{\mathbf{l}}\le\nu\\ |\mathbf{l}|>\mu}} (2^{p+1}c_p/a_-^p)^{j_{\mathbf{l}}}\cdot 2^{-|\mathbf{l}|p}\cdot (pj_{\mathbf{l}})!\cdot \prod_{j=1}^{j_{\mathbf{l}}} \rho_{m_j}^p \cdot \|\tilde{u}_M\|_{L^2(I^M)} \\
&\lesssim \sum_{\substack{\mathbf{l}\in\mathbb{N}^M\\ j_{\mathbf{l}}\le\nu\\ |\mathbf{l}|>\mu}} a^{(1+\kappa)j_{\mathbf{l}}^{1+\kappa}p}\cdot 2^{-|\mathbf{l}|p}\cdot \|\tilde{u}_M\|_{L^2(I^M)}, \qquad (41)
\end{aligned}$$

$\forall a\in(e^{-c_r},1)$ and with a constant depending on a,r,p.

Use counting argument in the r.h.s. of (41) and

Lemma B For any $t\in[0,1)$ and $j,L\in\mathbb{N}$ with $j\le L$ it holds

$$\sum_{n\ge 0}\binom{L+n}{j}t^n \le (L+1)^j(1-t)^{-j-1}. \qquad (42)$$

Use Lemma B with $t = 2^{-p}$ to get

$$\begin{aligned}
\sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ j_{\mathbf{l}}\le\nu \\ |\mathbf{l}|>\mu}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} &\lesssim \sum_{j=1}^{\nu} \binom{M}{j} a^{(1+\kappa)j^{1+\kappa}p} \cdot \sum_{l=\mu+1}^{\infty} \binom{l}{j} 2^{-pl} \cdot \|\tilde{u}_M\|_{L^2(I^M)} \\
&\le 2^{-p(\mu+1)} \cdot \sum_{j=1}^{\nu} \binom{M}{j} a^{(1+\kappa)j^{1+\kappa}p} \cdot (1-2^{-p})^{-j-1} \cdot (\mu+2)^j \cdot \|\tilde{u}_M\|_{L^2(I^M)} \\
&\lesssim 2^{-p(\mu+1)} \cdot \max_{0\le q\le\nu} \binom{M}{q} \cdot 2^{\nu+1} \cdot (\mu+2)^{\nu} \cdot \|\tilde{u}_M\|_{L^2(I^M)}, \qquad (43)
\end{aligned}$$

with a constant depending on a, r, p.

The r.h.s. of (43) can be further estimated as follows

$$\begin{aligned}
\sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ j_{\mathbf{l}}\le\nu \\ |\mathbf{l}|>\mu}} \|\tilde{u}_M^{\mathbf{l}}\|_{L^2(I^M)} &\lesssim 2^{-p(\mu+1)} \cdot (M+1)^{\nu} \cdot 2^{\nu} \cdot (\mu+2)^{\nu} \cdot \|\tilde{u}_M\|_{L^2(I^M)} \\
&= e^{-p(\mu+1)\log 2 + \nu(\log(M+1)+\log 2+\log(\mu+2))} \cdot \|\tilde{u}_M\|_{L^2(I^M)} \qquad (44)
\end{aligned}$$

We note that ν has been already chosen as in (32) with c_3 given by (40).

Choosing now μ as in (32), with

$$c_4 := \frac{c_r}{p \log 2} \tag{45}$$

we immediately see from (44) that the upper bound in (33) is matched. The proof of (33) is therefore complete.

Estimate of dimension of $\widehat{V}^{\mu,\nu}$: with μ, ν as in (32).

$$\begin{aligned} \dim \widehat{V}^{\mu,\nu} = \text{Card } \Lambda^{\mu,\nu} &= \sum_{q=0}^{\nu} \sum_{l=0}^{\mu} \binom{M}{q} \binom{l}{q} 2^l p \\ &\leq p(M+1)^\nu \sum_{q=0}^{\nu} \sum_{l=0}^{\mu} \binom{l}{q} 2^l \\ &\leq p(M+1)^\nu \sum_{l=0}^{\mu} (l+1)^\nu 2^l \leq p(M+1)^\nu \mu(\mu+1)^\nu 2^\mu \\ &= e^{\log p + \nu(\log(M+1) + \log(\mu+1)) + \mu \log 2} \end{aligned} \tag{46}$$

(34) follows then by using (32), (40) and (45) in (46). ■

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: p FEM -

Theorem 9 (Todor and Sc. 2005)
If C_a pw analytic exists $\widehat{V}_M$ *sparse* polynomial space in M variables such that

i. ($\mathcal{P}$): The following approximation property holds,

$$\begin{aligned}
\inf_{v\in\widehat{V}_M} \|\tilde{u}_M - v\|_{H_0^1(D)\otimes L^\infty(I^M)} &\lesssim \exp(-cM^{1/d}) \\
\widehat{N} := \dim \widehat{V}_M &\lesssim \exp(\widehat{c}M^{1/(d+1)}\log(M)) \\
\inf_{v\in\widehat{V}_M} \|\tilde{u}_M - v\|_{H_0^1(D)\otimes L^\infty(I^M)} &\lesssim \widehat{N}^{-k} \qquad \forall k > 0
\end{aligned}$$

ii. ($\mathcal{B}$): There exists a basis of $\widehat{V}_M$ such that the stiffness matrix of the bvp with stochastic data is well-conditioned and sparse in the stochastic variable y (at most $O(M)$ nontrivial entries in each row)

Reduction to high dimensional deterministic bvp

- stochastic semi-discretization: p FEM -

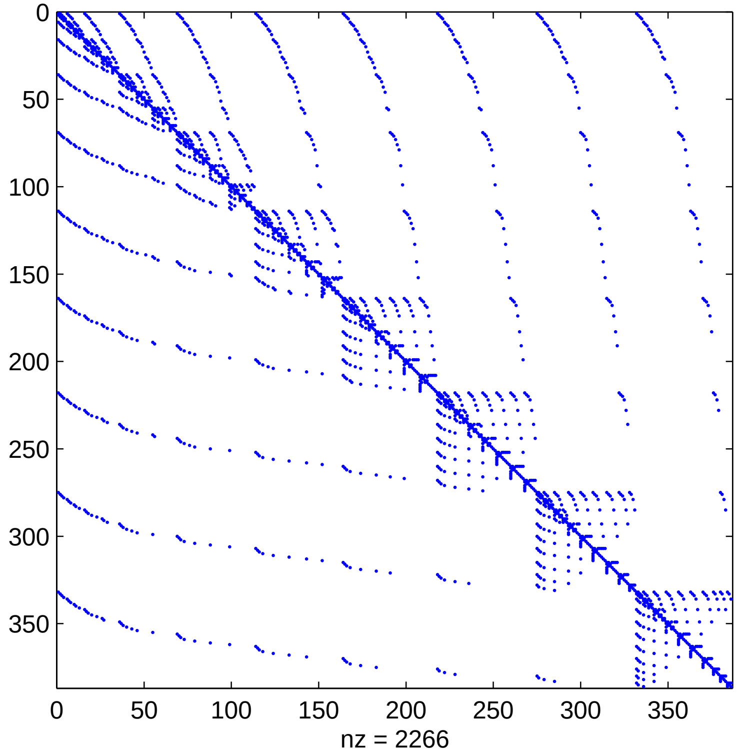
Comparison of required number of dof's ($d = 2$)

	M=5	M=10	M=20	M=30	M=50
uniform polynomial degree	7776	2.5e+10	2.7e+26	5.5e+44	2.3e+85
adapted polynomial degree	60	19200	3.0e+09	6.8e+14	5.8e+25
sparse polynomial space $\widehat{V}_M$	16	143	3401	38883	1815763

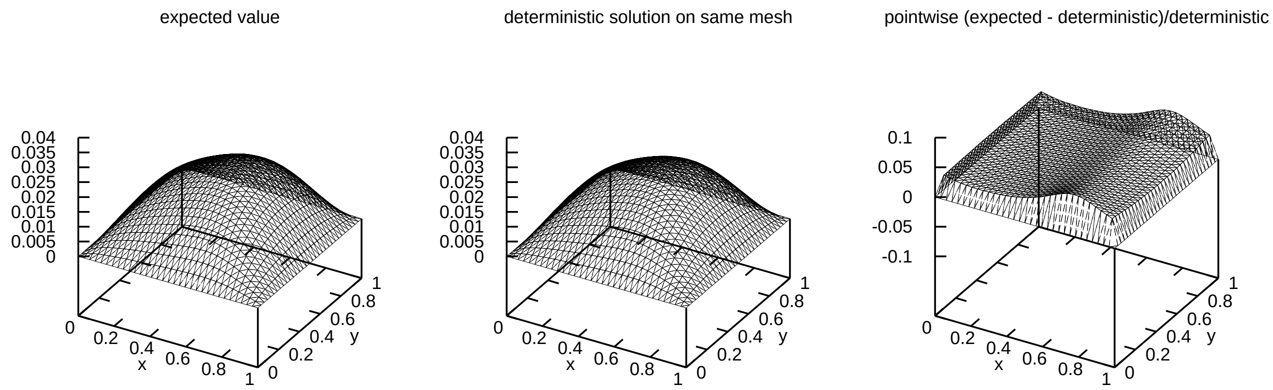
Reduction to high dimensional deterministic bvp

- stochastic semi-discretization -

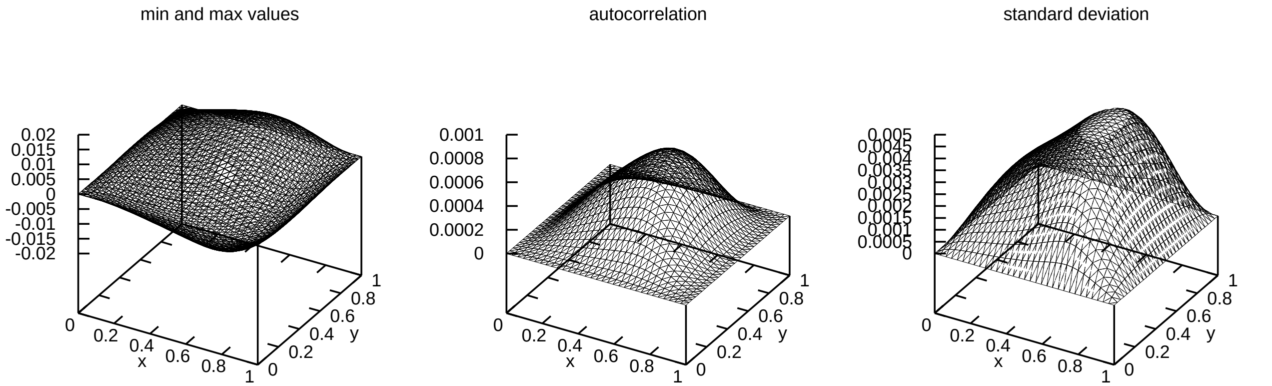
Sparsity pattern w.r.t. y of stochastic moment matrix



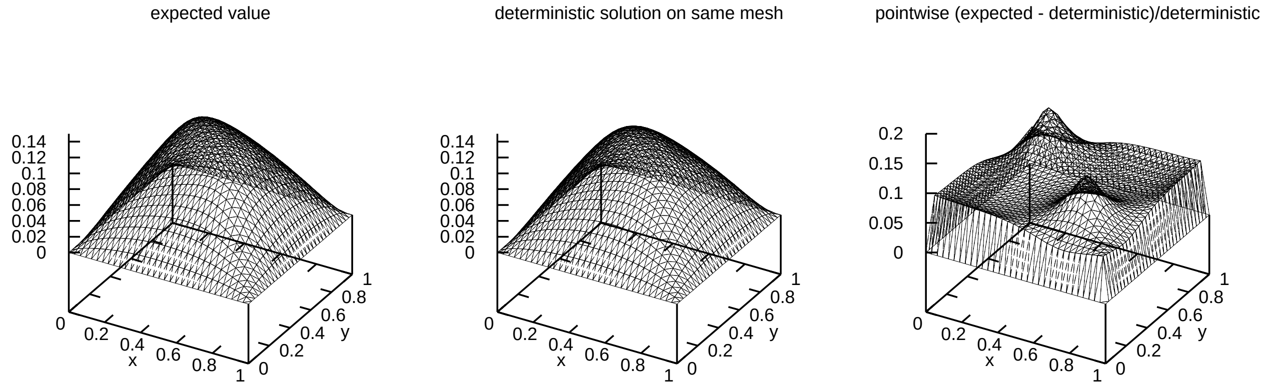
Numerical experiments



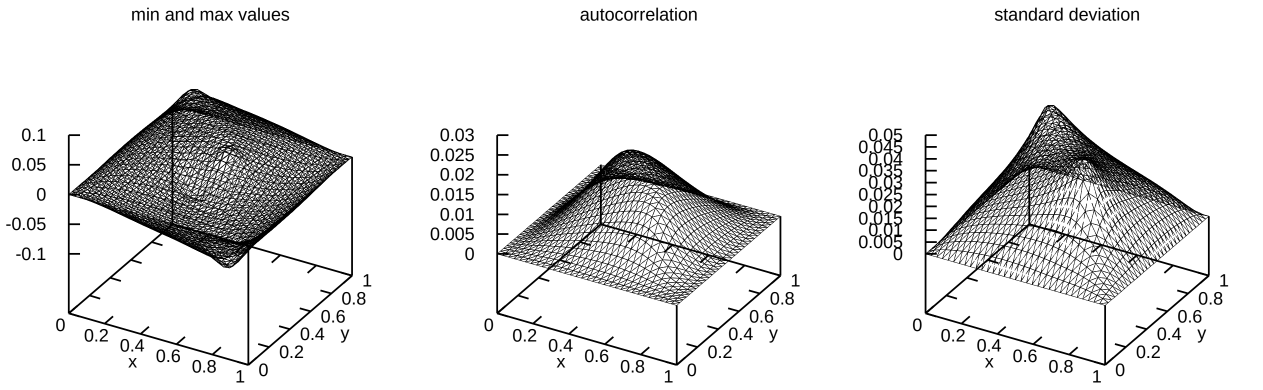
$E_a = 3+\sin(2*pi*x_1)$, $C_a = e^{-|x-y|^2}$, f = 1, l = 5 (2048 Elements), r = 4,2,2,1,1,1,1,1,1,1,1,1,1 (46080 det. Problems)



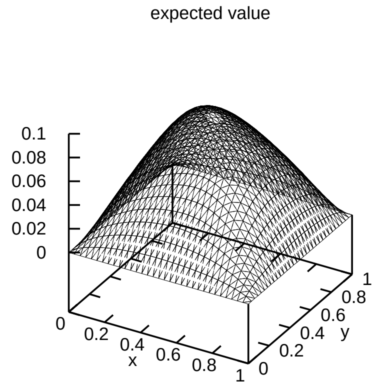
Numerical experiments

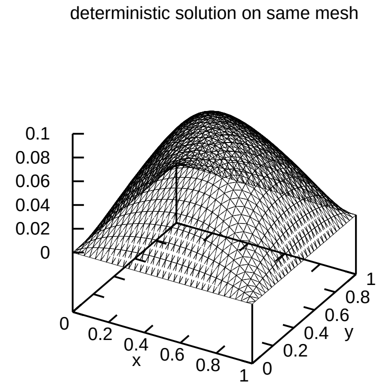


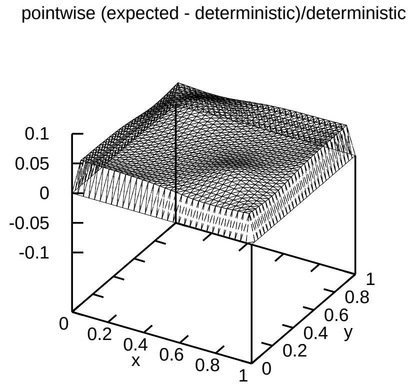
E_a=2.14+sin(2*pi*x_1)sin(2*pi*x_2), $C_a=e^{-4*|x-y|^2}$, f=3+x_1, l = 5 (2048 El.), r = 3,2,2,2,1,1,1,1, (1728 det. Prob.)



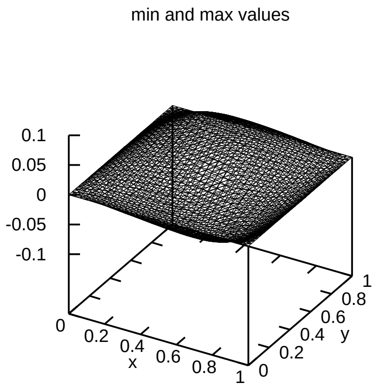
Numerical experiments

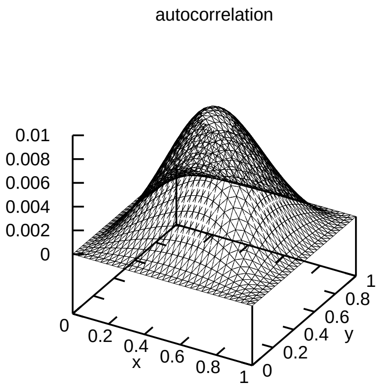


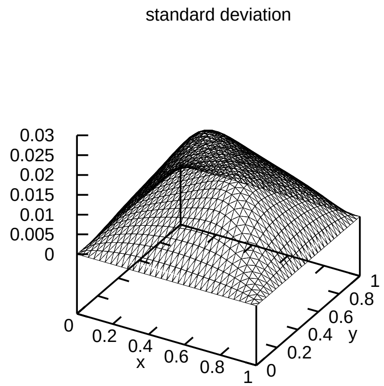




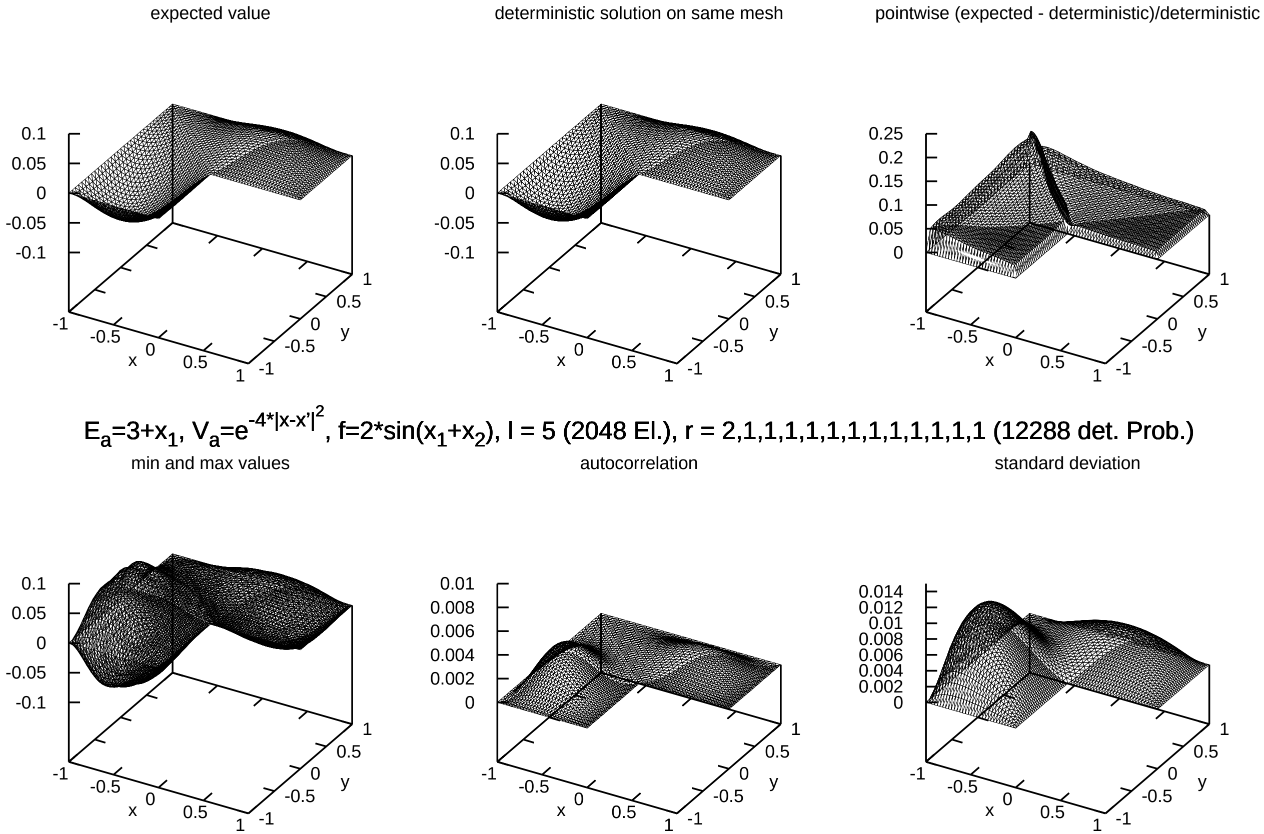
$E_a=2.7+\sin(2*pi*x_1)\sin(2*pi*x_2)$, $C_a=e^{-4*|x-y|^2}$, $f=3+x_1$, l = 5 (2048 El.), r = 3,2,2,2,1,1,1,1,1,1,1,1 (27648 det. Prob.)







Numerical experiments



Numerical experiments