A Survey of Numerical Methods for Computational Aeroacoustics

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Part I Finite Difference Schemes

Part II Finite Element and Discontinuous Galerkin Methods

Boundary Element Method

Method of Lines

Partial differential equations
$$\frac{\partial u}{\partial t} = F(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \cdots)$$

$\Downarrow$ Spatial discretization

Semi-discrete equations
$$\frac{\partial u_h}{\partial t} = F(u_h, D_x u_h, D_y u_h, \cdots)$$

$\Downarrow$ Temporal discretization

Fully discrete equations
$$u_h^{n+1} = u_h^n + R(u_h^n, u_h^{n-1}, \cdots)$$

Question: how to analyze each element of the scheme in the wavenumber/frequency space

Part I

- ⧫ Spatial discretization by finite difference schemes
- ⧫ Time integration schemes
- ⧫ Boundary closures
- ⧫ Numerical dissipation models
- ⧫ Overset grid for complex geometries

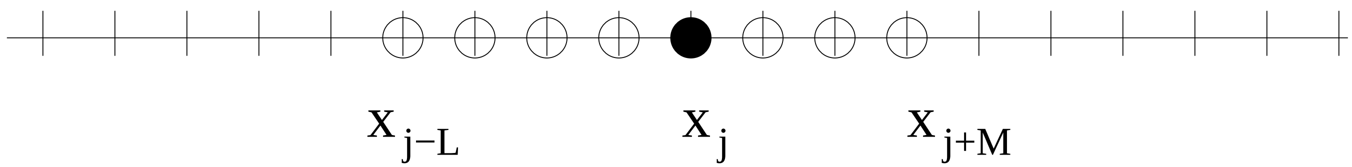
Discretization of spatial derivatives by finite differences

1. Explicit finite difference schemes

2. Compact finite difference schemes (implicit)

Discretization of spatial derivatives

1. Explicit finite difference schemes



$$\left(\frac{\partial u}{\partial x}\right)_j \approx \frac{1}{\Delta x} \sum_{\ell=-L}^{M} a_\ell u_{j+\ell}$$

Central: $L = M$ Upwind: $L > M$

Consider the continuous model

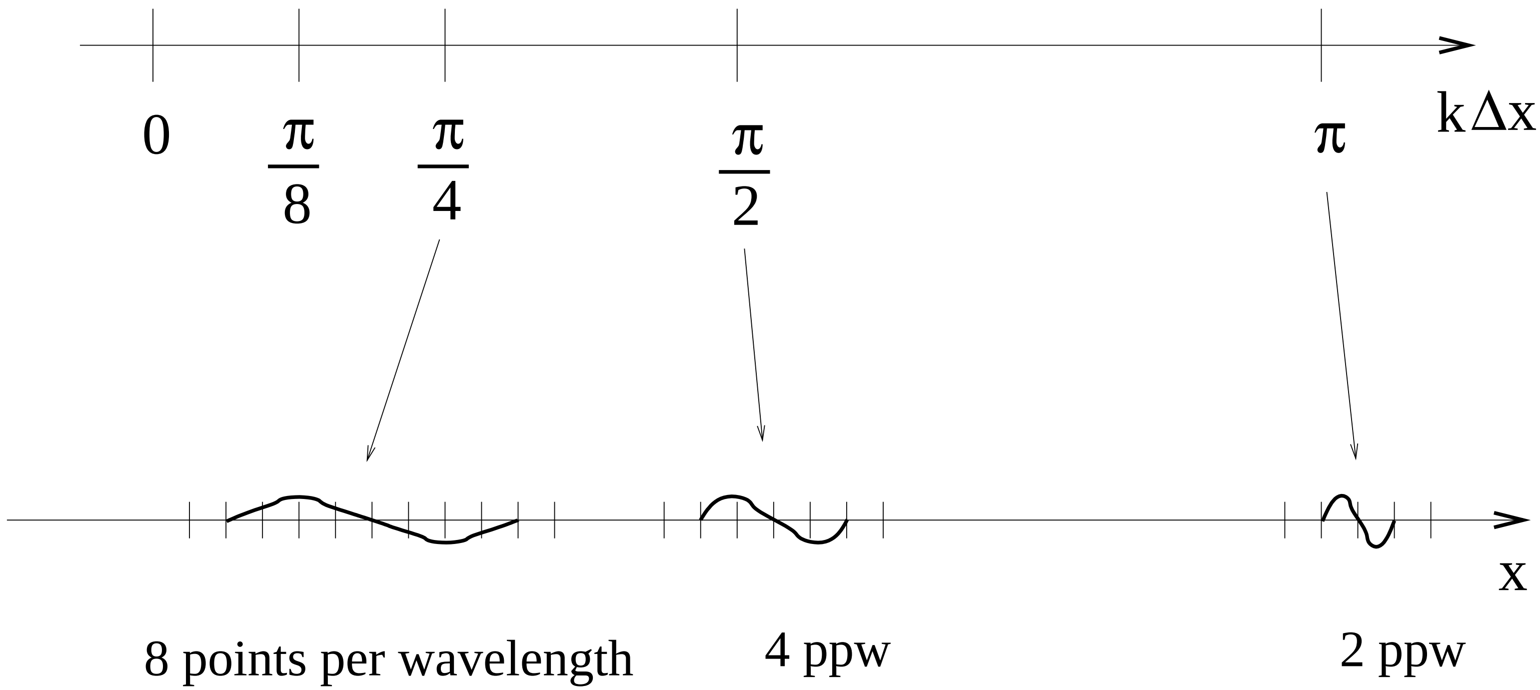
$$\frac{\partial u}{\partial x}(x) \approx \frac{1}{\Delta x} \sum_{\ell=-L}^{M} a_\ell u(x + \ell \Delta x)$$

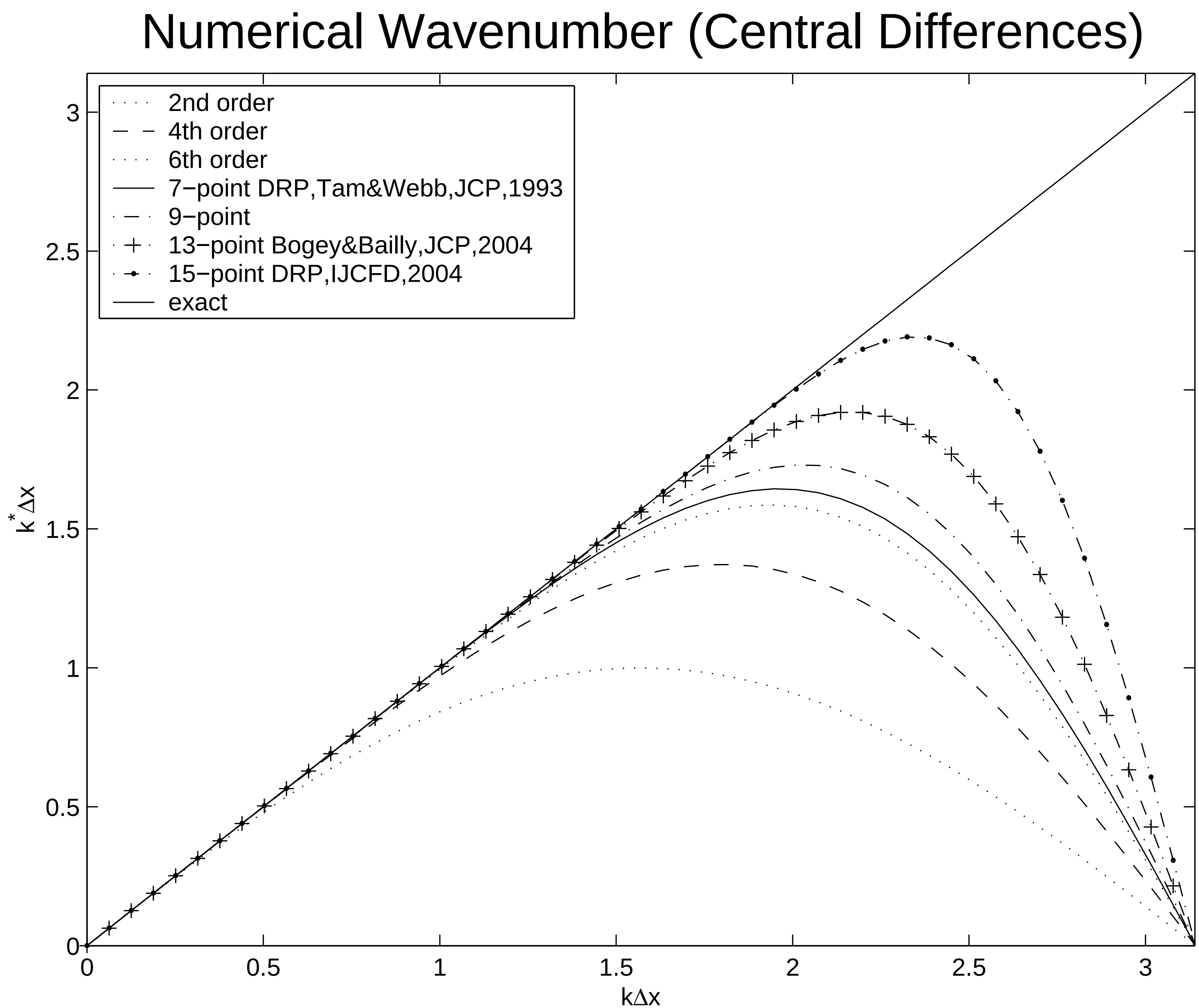
Fourier transform gives

$$ik \approx \frac{1}{\Delta x} \sum_{\ell=-L}^{M} a_\ell e^{i\ell k \Delta x}$$

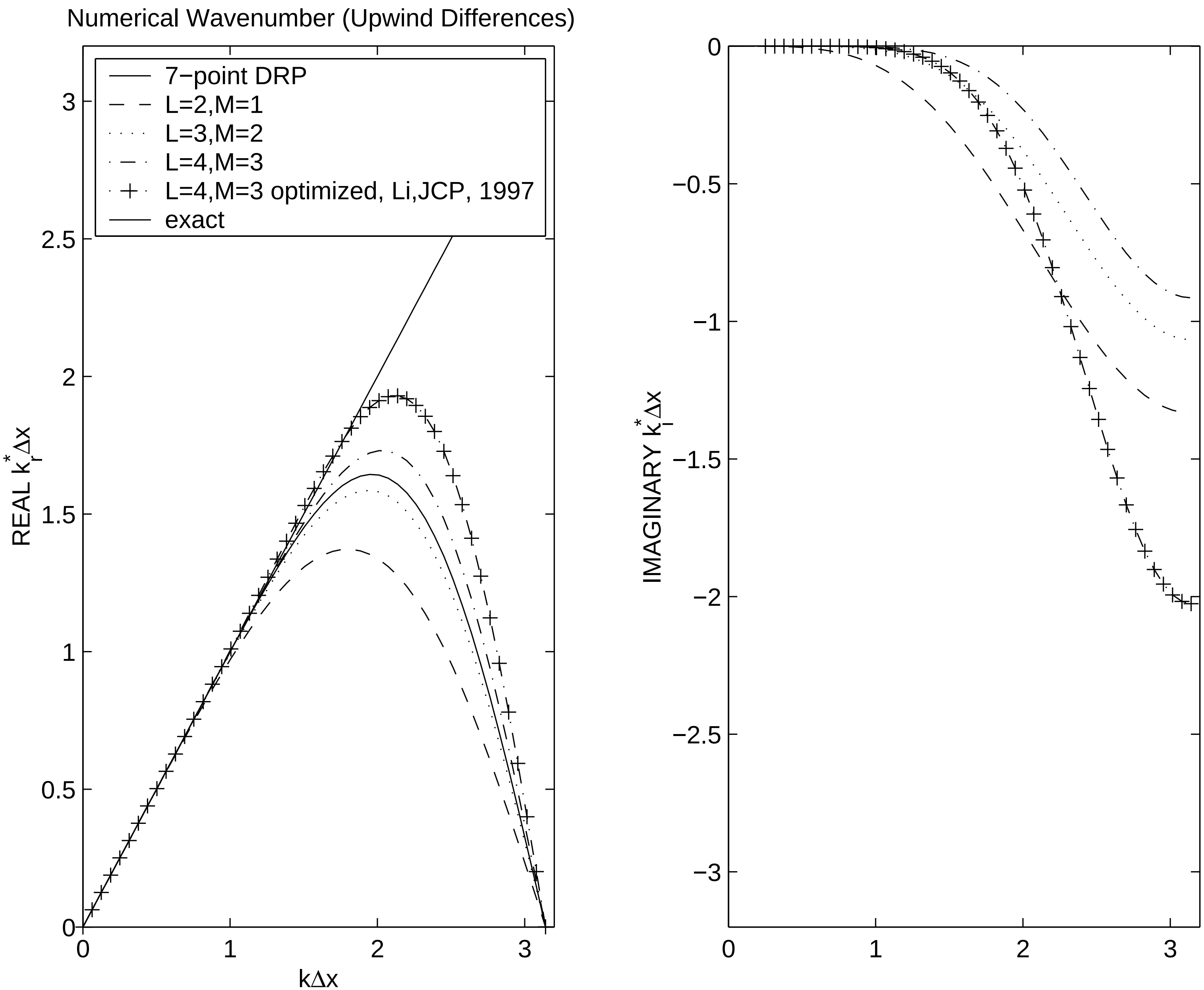
$k^* \Delta x = -i \sum_{\ell=-L}^{M} a_\ell e^{i\ell k \Delta x}$ is the intrinsic numerical wave number

Non-dimensional wave number $k\Delta x$

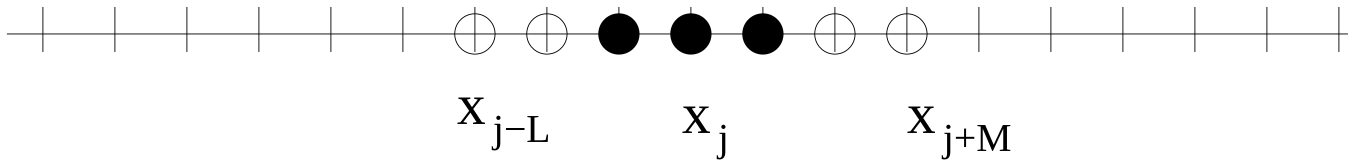



Numerical Wavenumber (Central Differences)
2nd order
4th order
6th order
7−point DRP,Tam&Webb,JCP,1993
9−point
13−point Bogey&Bailly,JCP,2004
15−point DRP,IJCFD,2004
exact
k*Δx
kΔx
0
0.5
1
1.5
2
2.5
3

Numerical Wavenumber (Upwind differences)



2. Compact finite difference schemes (implicit)



$$\sum_{\ell=-K}^{N} c_\ell \left(\frac{\partial u}{\partial x}\right)_{j+\ell} \approx \frac{1}{\Delta x} \sum_{\ell=-L}^{M} a_\ell u_{j+\ell} \quad \text{(a system of equations)}$$

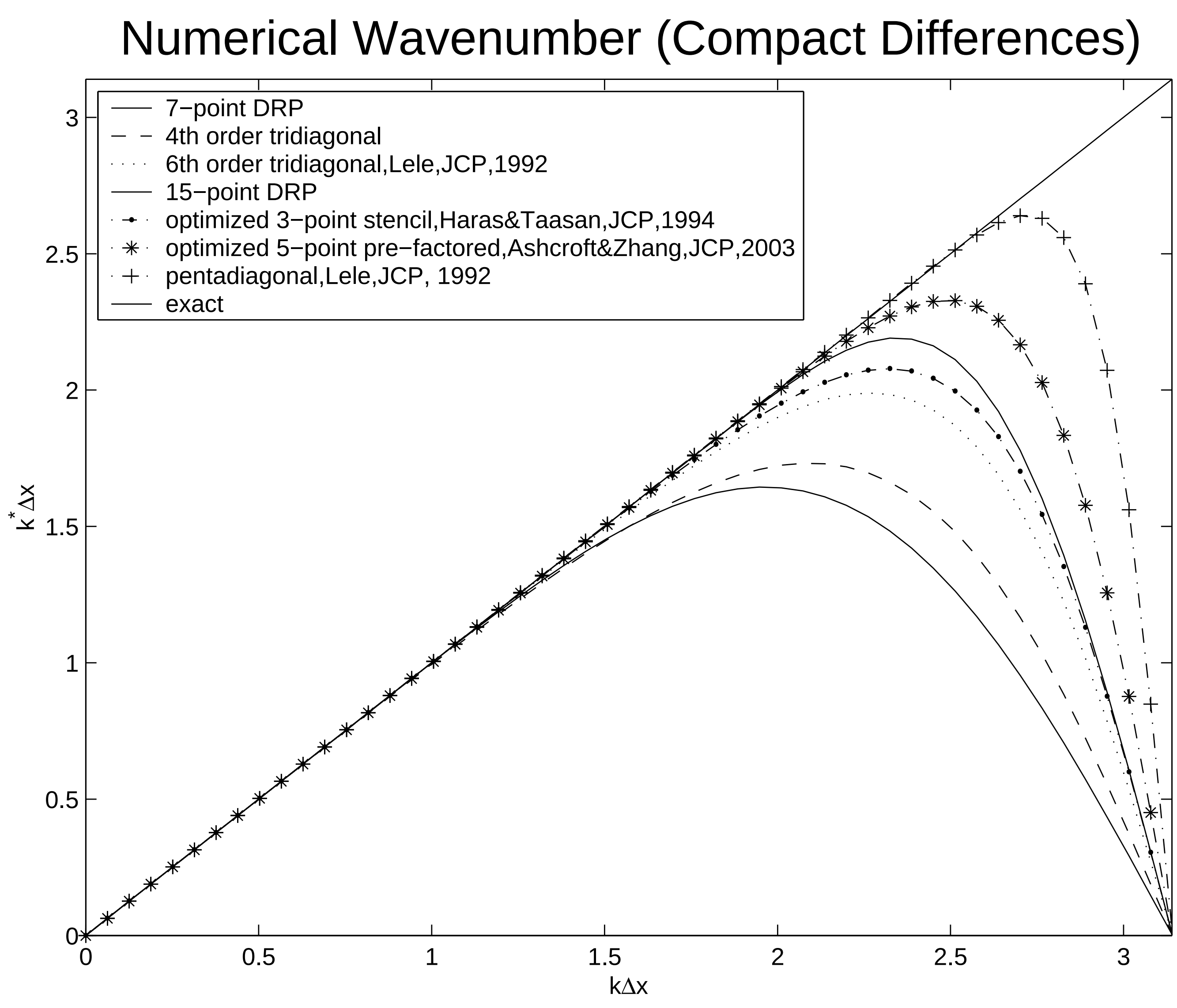
Consider the continuous model

$$\sum_{\ell=-K}^{N} c_\ell \frac{\partial u}{\partial x}(x+\ell\Delta) \approx \frac{1}{\Delta x} \sum_{\ell=-L}^{M} a_\ell u(x+\ell\Delta x)$$

Fourier transform gives

$$ik \sum_{\ell=-K}^{N} c_\ell e^{i\ell k\Delta} \approx \frac{1}{\Delta x} \sum_{\ell=-L}^{M} a_\ell e^{i\ell k\Delta x}$$

$$k^*\Delta x = -i\frac{\displaystyle\sum_{\ell=-L}^{M} a_\ell e^{i\ell k\Delta x}}{\displaystyle\sum_{\ell=-K}^{N} c_\ell e^{i\ell k\Delta}} \text{ is the numerical wave number}$$



Numerical Wavenumber (Compact Differences)
7−point DRP
4th order tridiagonal
6th order tridiagonal,Lele,JCP,1992
15−point DRP
optimized 3−point stencil,Haras&Taasan,JCP,1994
optimized 5−point pre−factored,Ashcroft&Zhang,JCP,2003
pentadiagonal,Lele,JCP, 1992
exact
k*Δx
kΔx
0
0.5
1
1.5
2
2.5
3

Time discretization

1. Runge-Kutta schemes (multi-stage)

2. Adam-Bashforth schemes (multi-step)

Time discretization

Semi-discrete equation: $\dfrac{\partial u_h}{\partial t} = F(u_h)$

1. Runge-Kutta schemes (multi-stage)

$$u_h^{n+1} = u_h^n + \sum_{i=1}^{p} w_i K_i$$

$$K_i = \Delta t\ F(u_h^n + \sum_{j=1}^{i-1} \beta_{ij} K_j)$$

A low-storage linear implementation

$$K_1 = \Delta t\ F(u_h^n)$$

$$K_2 = \Delta t\ F(u_h^n + \beta_2 K_1)$$

$$\cdots\cdots$$

$$K_p = \Delta t\ F(u_h^n + \beta_p K_{p-1})$$

$$u_h^{n+1} = u_h^n + K_p$$

What do K_1, K_2, $\cdots\cdots$ mean?

$$K_1 = \Delta t\, \frac{\partial u_h}{\partial t}$$

$$K_2 = \Delta t\, \frac{\partial u_h}{\partial t} + \beta_2 \Delta t^2 \frac{\partial^2 u_h}{\partial t^2}$$

$$K_3 = \Delta t\, \frac{\partial u_h}{\partial t} + \beta_3 \Delta t^2 \frac{\partial^2 u_h}{\partial t^2} + \beta_3 \beta_2 \Delta t^3 \frac{\partial^3 u_h}{\partial t^3}$$

$$\cdots\cdots$$

$$u_h^{n+1} = u_h^n + \Delta t\, \frac{\partial u_h^n}{\partial t} + \beta_p \Delta t^2 \frac{\partial^2 u_h^n}{\partial t^2} + \beta_p \beta_{p-1} \Delta t^3 \frac{\partial^3 u_h^n}{\partial t^3} \cdots + \beta_p \beta_{p-1} \cdots \beta_2 \Delta t^p \frac{\partial^p u_h^n}{\partial t^p}$$

Continuous equations:

$$u(t + \Delta t) \approx u(t) + c_1 \Delta t\, \frac{\partial u}{\partial t}(t) + c_2 \Delta t^2 \frac{\partial^2 u}{\partial t^2} + \cdots c_p \Delta t^p \frac{\partial^p u}{\partial t^p}$$

Laplace transform:

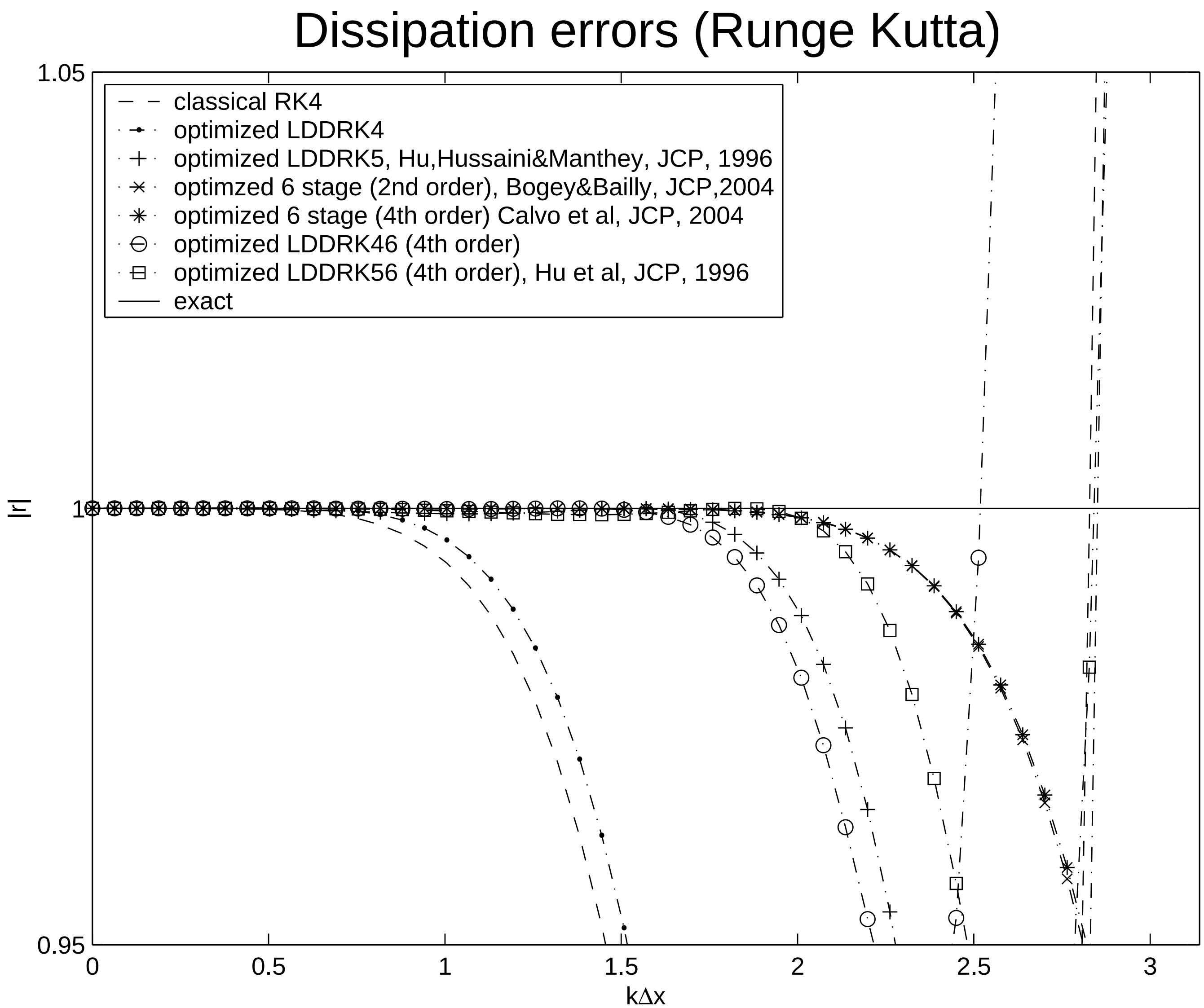
$$e^{-i\omega\Delta t} \approx 1 + c_1(-i\omega\Delta t) + c_2(-i\omega\Delta t)^2 + \cdots\cdots c_p(-i\omega\Delta t)^p$$

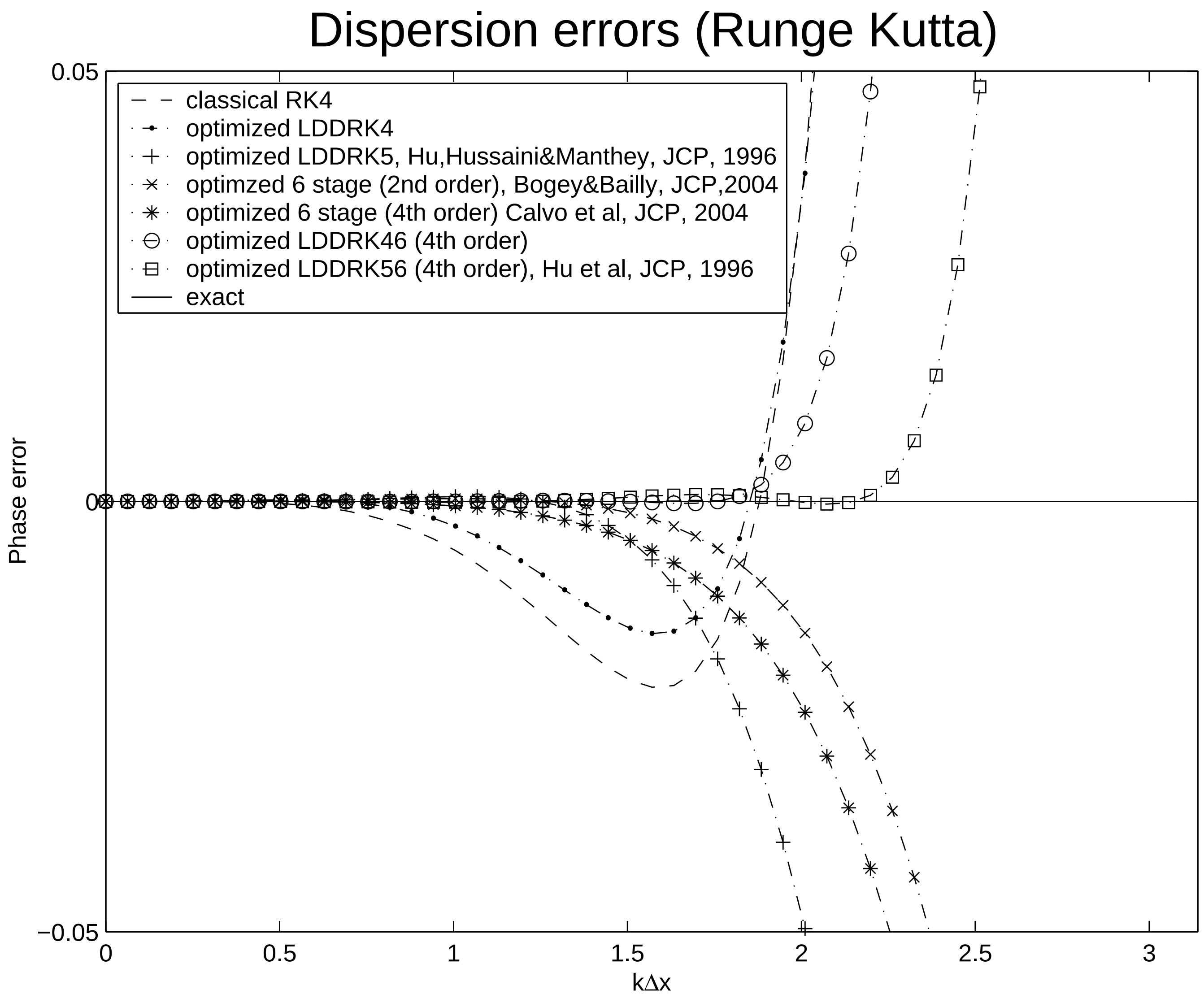
Amplification factor

$$r(\omega\Delta t) = 1 + c_1(-i\omega\Delta t) + c_2(-i\omega\Delta t)^2 + \cdots\cdots c_p(-i\omega\Delta t)^p \approx e^{-i\omega\Delta t}$$

$$\text{Dissipation Error (amplitude)} \;= 1 - |r(\omega\Delta t)|$$

$$\text{Dispersion Error (phase)} \;= i \ln[\frac{r(\omega\Delta t)}{e^{-i\omega\Delta t}}]$$


Dissipation errors (Runge Kutta)
1.05
1
0.95
|r|
0
0.5
1
1.5
2
2.5
3
kΔx
classical RK4
optimized LDDRK4
optimized LDDRK5, Hu,Hussaini&Manthey, JCP, 1996
optimzed 6 stage (2nd order), Bogey&Bailly, JCP,2004
optimized 6 stage (4th order) Calvo et al, JCP, 2004
optimized LDDRK46 (4th order)
optimized LDDRK56 (4th order), Hu et al, JCP, 1996
exact



Dispersion errors (Runge Kutta)
classical RK4
optimized LDDRK4
optimized LDDRK5, Hu,Hussaini&Manthey, JCP, 1996
optimzed 6 stage (2nd order), Bogey&Bailly, JCP,2004
optimized 6 stage (4th order) Calvo et al, JCP, 2004
optimized LDDRK46 (4th order)
optimized LDDRK56 (4th order), Hu et al, JCP, 1996
exact
Phase error
0.05
0
−0.05
0
0.5
1
1.5
2
2.5
3
kΔx

2. Adam-Bashforth schemes (multi-step)

$$u_h^{n+1} = u_h^n + \Delta t \sum_{j=0}^{N} b_j \left(\frac{du_h}{dt}\right)^{n-j}$$

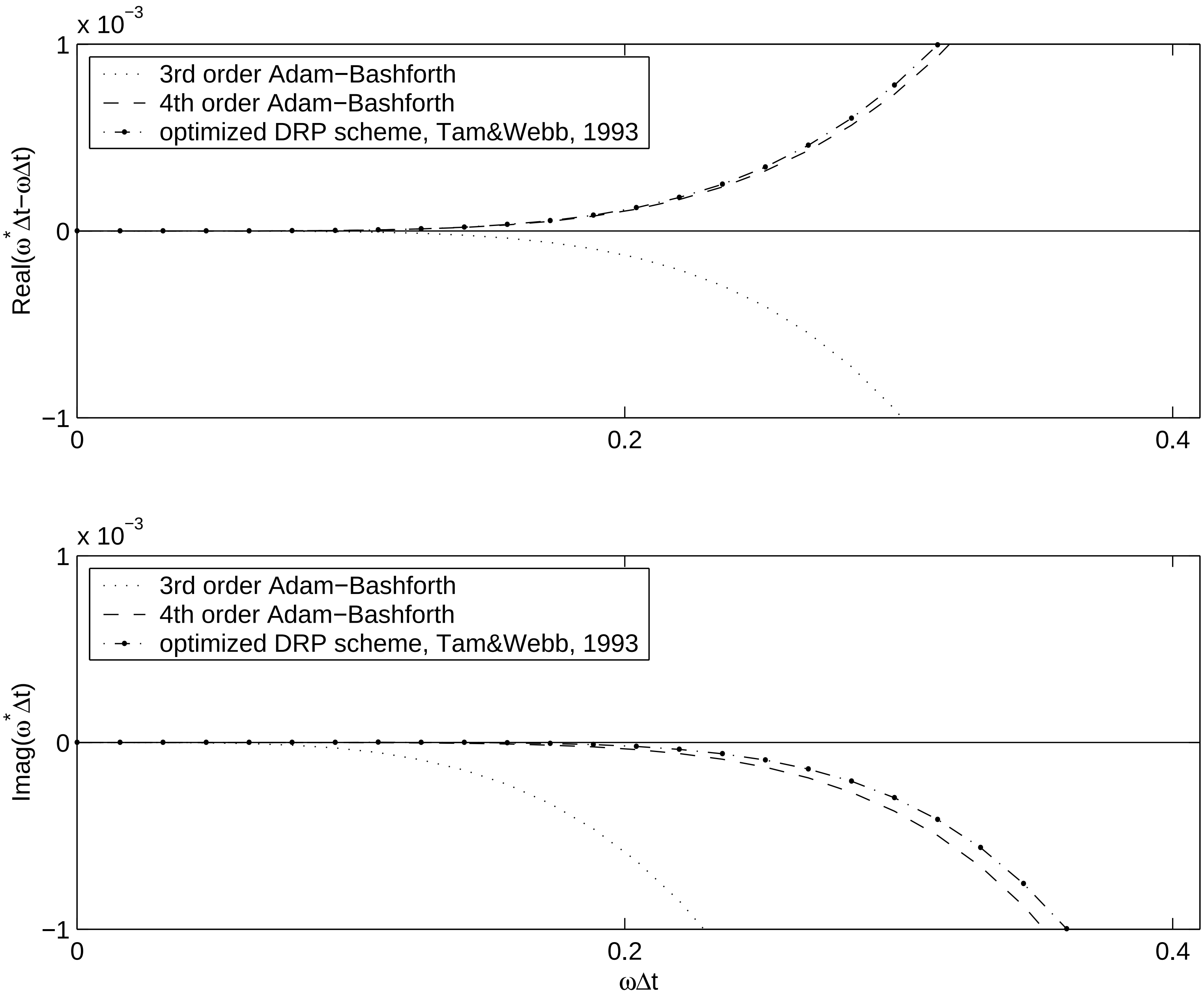
continuous model:

$$u_h(t + \Delta t) \approx u_h(t) + \Delta t \sum_{j=0}^{N} b_j \frac{du_h}{dt}(t - j\Delta t)$$

Laplace transform:

$$e^{-i\omega\Delta t} \approx 1 + \Delta t(-i\omega) \sum_{j=0}^{N} b_j e^{ij\omega\Delta t}$$

$$\omega^* \Delta t = i \frac{e^{-i\omega\Delta t} - 1}{\sum_{j=0}^{N} b_j e^{ij\omega\Delta t}} \text{ is the numerical frequency}$$

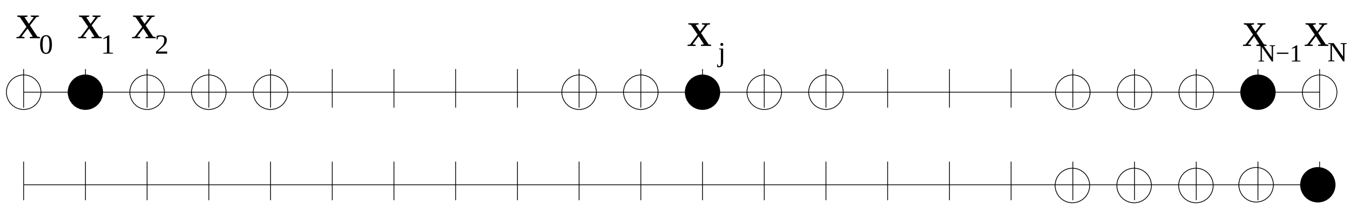


x 10^−3
1
0
−1
0
0.2
0.4
Real(ω*Δt−ωΔt)
3rd order Adam−Bashforth
4th order Adam−Bashforth
optimized DRP scheme, Tam&Webb, 1993
x 10^−3
Imag(ω*Δt)
ωΔt

Boundary closure schemes

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad a \leq x \leq b$$

$$\text{B.C. : } u(a,t) = g(t)$$



$$\left(\frac{\partial u}{\partial x}\right)_1 = \frac{1}{\Delta x}\left(-\frac{1}{4}u_0 - \frac{5}{6}u_1 + \frac{3}{2}u_2 - \frac{1}{2}u_3 + \frac{1}{12}u_4\right) \text{ (4th order)}$$

$$\left(\frac{\partial u}{\partial x}\right)_j = \frac{1}{\Delta x}\left(\frac{1}{12}u_{j-2} - \frac{8}{12}u_{j-1} + \frac{8}{12}u_{j+1} - \frac{1}{12}u_{j+2}\right) \text{ (4th order)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{N-1} = \frac{1}{\Delta x}\left(-\frac{1}{12}u_{N-4} + \frac{1}{2}u_{N-3} - \frac{3}{2}u_{N-2} + \frac{5}{6}u_{N-1} + \frac{1}{4}u_N\right) \text{ (4th order)}$$

$$\left(\frac{\partial u}{\partial x}\right)_N = \frac{1}{\Delta x}\left(\frac{1}{4}u_{N-4} - \frac{4}{3}u_{N-3} + 3u_{N-2} - 4u_{N-1} + \frac{25}{12}u_N\right) \text{ (4th order)}$$

Ref[1]: Strikwerda, JCP, Vol. 34, 94-107, 1980

Ref[2]: Carpenter,Gottlieb & Abarbanel, JCP, Vol 108, 272-295, 1993

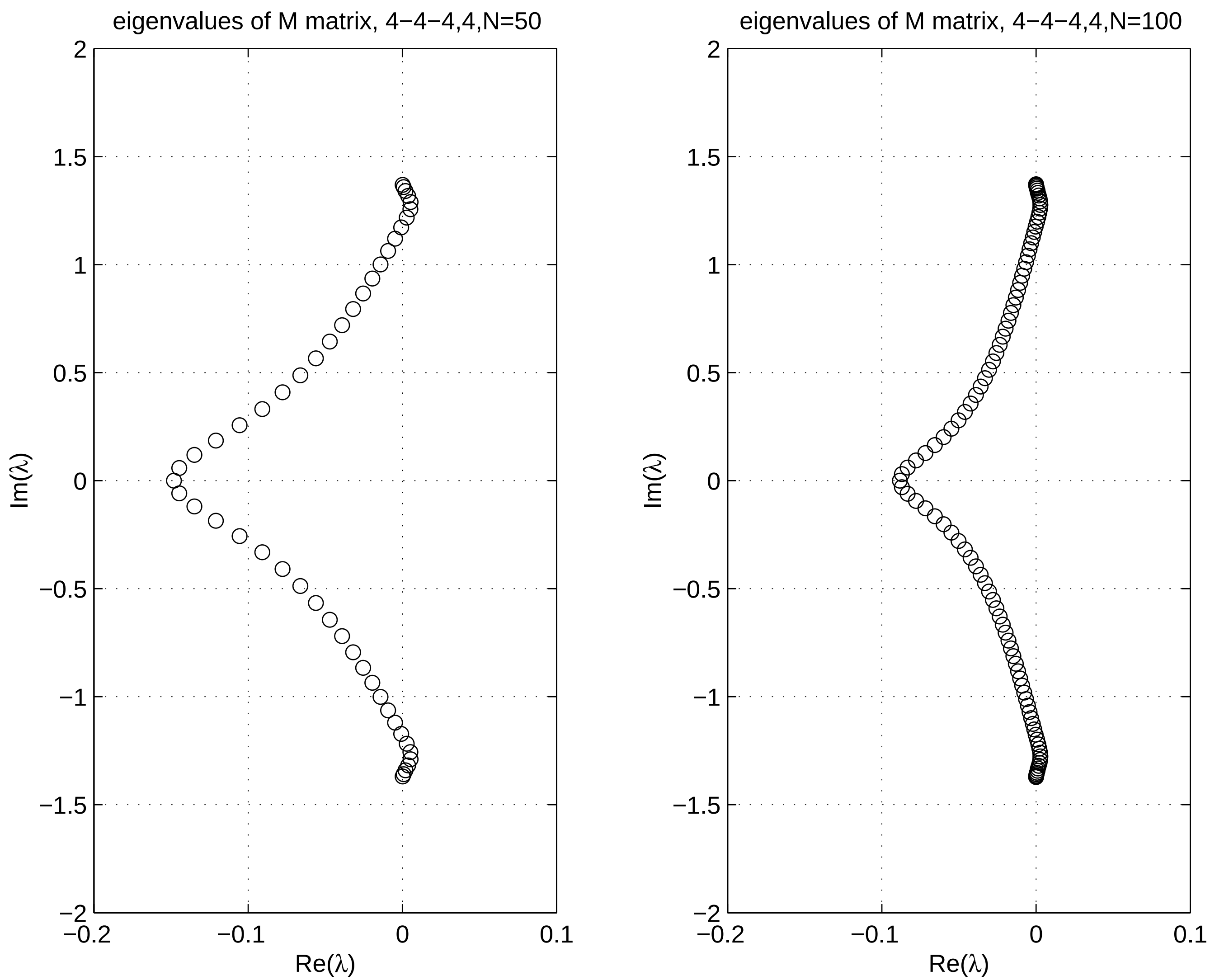
Semi-discrete equation (matrix form)

$$\frac{d\mathbf{u}}{dt}+\frac{1}{\Delta x}\begin{bmatrix} -\frac{5}{6} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{12} & \cdots & 0 & 0 & 0 & 0 & 0 \\ -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & \\ & \cdots & \frac{1}{12} & -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} & \cdots & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{12} & \frac{1}{2} & -\frac{3}{2} & \frac{5}{6} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \\ \\ \\ u_{N-1} \\ u_N \end{bmatrix}+\frac{1}{\Delta x}\begin{bmatrix} -\frac{1}{4}g(t) \\ \frac{1}{12}g(t) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}=0$$

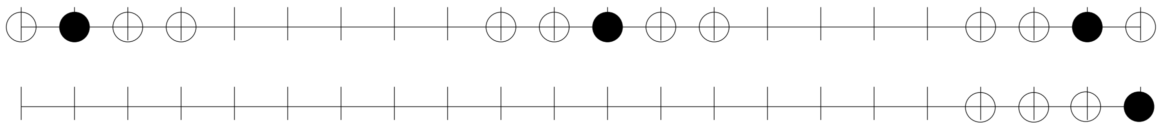
$$\frac{d\mathbf{u}}{dt}=\mathbf{M}\mathbf{u}+\mathbf{g}$$

For stability, we require that $Re\{\text{eigenvalue}(\mathbf{M})\} \leq 0$.

Eigenvalues of matrix $\mathbf{M}$



Boundary closure schemes



$$\left(\frac{\partial u}{\partial x}\right)_1 = \frac{1}{\Delta x}(-\frac{1}{3}u_0 - \frac{1}{2}u_1 + u_2 - \frac{1}{6}u_3) \text{ (3rd order)}$$

$$\left(\frac{\partial u}{\partial x}\right)_j = \frac{1}{\Delta x}(\frac{1}{12}u_{j-2} - \frac{8}{12}u_{j-1} + \frac{8}{12}u_{j+1} - \frac{1}{12}u_{j+2}) \text{ (4th order)}$$

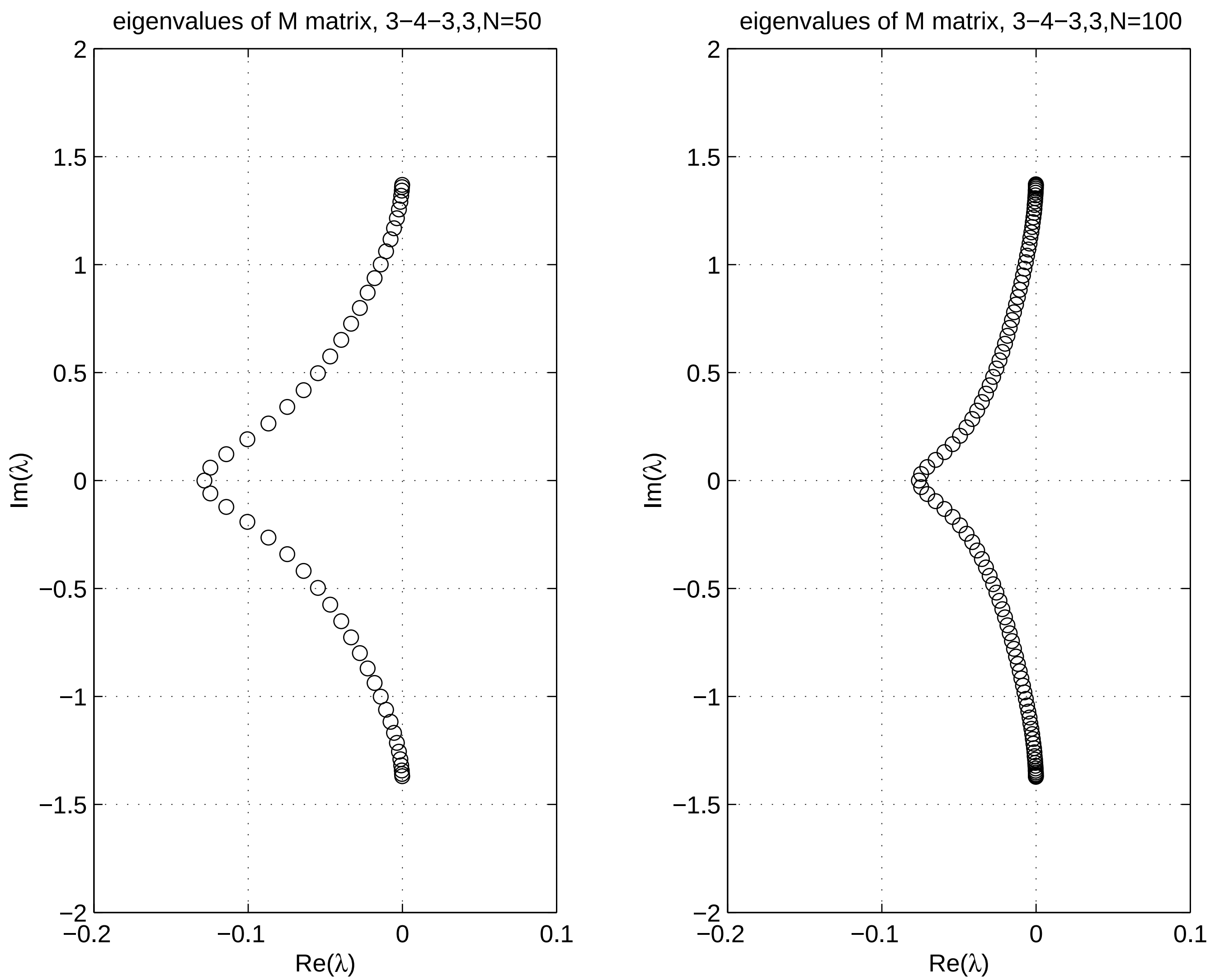
$$\left(\frac{\partial u}{\partial x}\right)_{N-1} = \frac{1}{\Delta x}(\frac{1}{6}u_{N-3} - u_{N-2} + \frac{1}{2}u_{N-1} + \frac{1}{3}u_N) \text{ (3rd order)}$$

$$\left(\frac{\partial u}{\partial x}\right)_N = \frac{1}{\Delta x}(-\frac{1}{3}u_{N-3} + \frac{3}{2}u_{N-2} - 3u_{N-1} + \frac{11}{6}u_N) \text{ (3rd order)}$$

Semi-discrete equation (matrix form)

$$\frac{d\mathbf{u}}{dt} + \frac{1}{\Delta x}\begin{bmatrix} -\frac{1}{2} & 1 & -\frac{1}{6} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & \\ & \cdots & \frac{1}{12} & -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} & \cdots & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{6} & -1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{3} & \frac{3}{2} & -3 & \frac{11}{6} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \\ \\ \\ u_{N-1} \\ u_N \end{bmatrix} + \frac{1}{\Delta x}\begin{bmatrix} -\frac{1}{3}g(t) \\ \frac{1}{12}g(t) \\ 0 \\ \\ \\ 0 \\ 0 \end{bmatrix} = 0$$

Eigenvalues of matrix M (3rd order closure)



Numerical damping models

Why numerical damping

1. Eliminate unresolved short waves
2. Improve stability
3. Desirable for central schemes

Objective of damping:

Eliminate short unresolved waves while keep resolved waves intact

Methods of numerical damping

1. Explicit filters
2. Implicit (compact) filters
3. Artificial damping term

1. Explicit filters

$$\bar{u}_j = u_j - \sum_{\ell=-N}^{N} d_\ell u_{j+\ell}$$

Continuous model:

$$\bar{u}(x) = u(x) - \sum_{\ell=-N}^{N} d_\ell u(x + \ell\Delta x)$$

Fourier transform:

$$\bar{u}(k) = (1 - \sum_{\ell=-N}^{N} d_\ell e^{i\ell\Delta x k})u(k)$$

Nth-order filter:

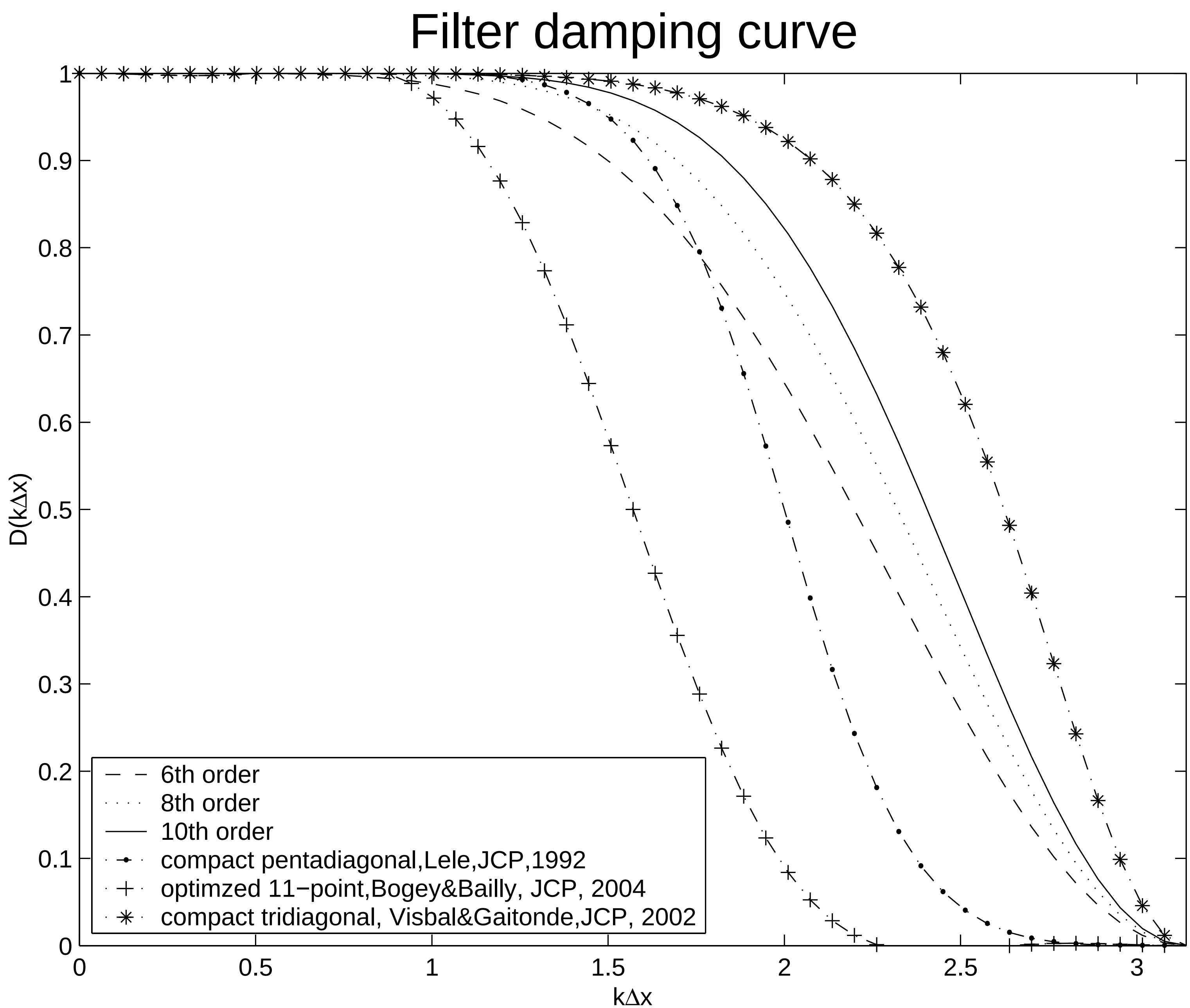
$$1 - \sum_{\ell=-N}^{N} d_\ell e^{i\ell k\Delta x} = 1 - \sin^N(\frac{k\Delta x}{2})$$

2. Implicit (compact) filters

$$\bar{u}_j + \sum_{\ell=1}^{N} a_\ell(\bar{u}_{j-\ell} + \bar{u}_{j+\ell}) = d_0 u_j + \sum_{\ell=1}^{M} d_\ell(u_{j-\ell} + u_{j+\ell})$$

Wavenumber space:

$$\bar{u}(k) = \frac{d_0 + 2\sum_{j=1}^{3} d_j \cos(jk\Delta x)}{1 + 2\sum_{j=1}^{2} a_j \cos(jk\Delta x)} u(k)$$



Filter damping curve
D(kΔx)
kΔx
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
6th order
8th order
10th order
compact pentadiagonal,Lele,JCP,1992
optimzed 11−point,Bogey&Bailly, JCP, 2004
compact tridiagonal, Visbal&Gaitonde,JCP, 2002

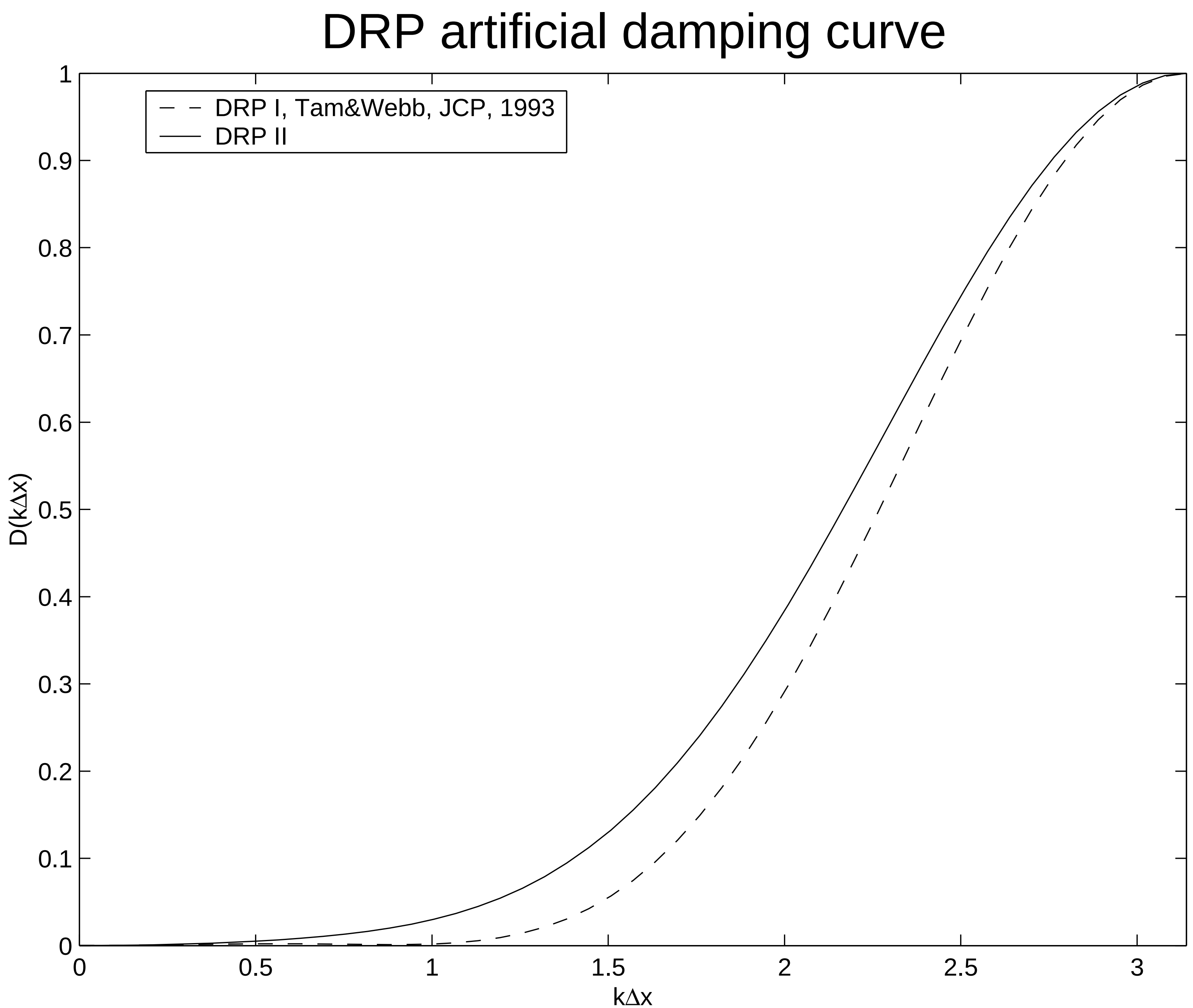
3. Artificial damping

$$\frac{du_j}{dt} + \cdots = -\frac{R_{\text{stancil}}}{\Delta x}[d_0 u_j + \sum_{\ell=1}^{3} d_\ell (u_{j-\ell} + u_{j+\ell})]$$

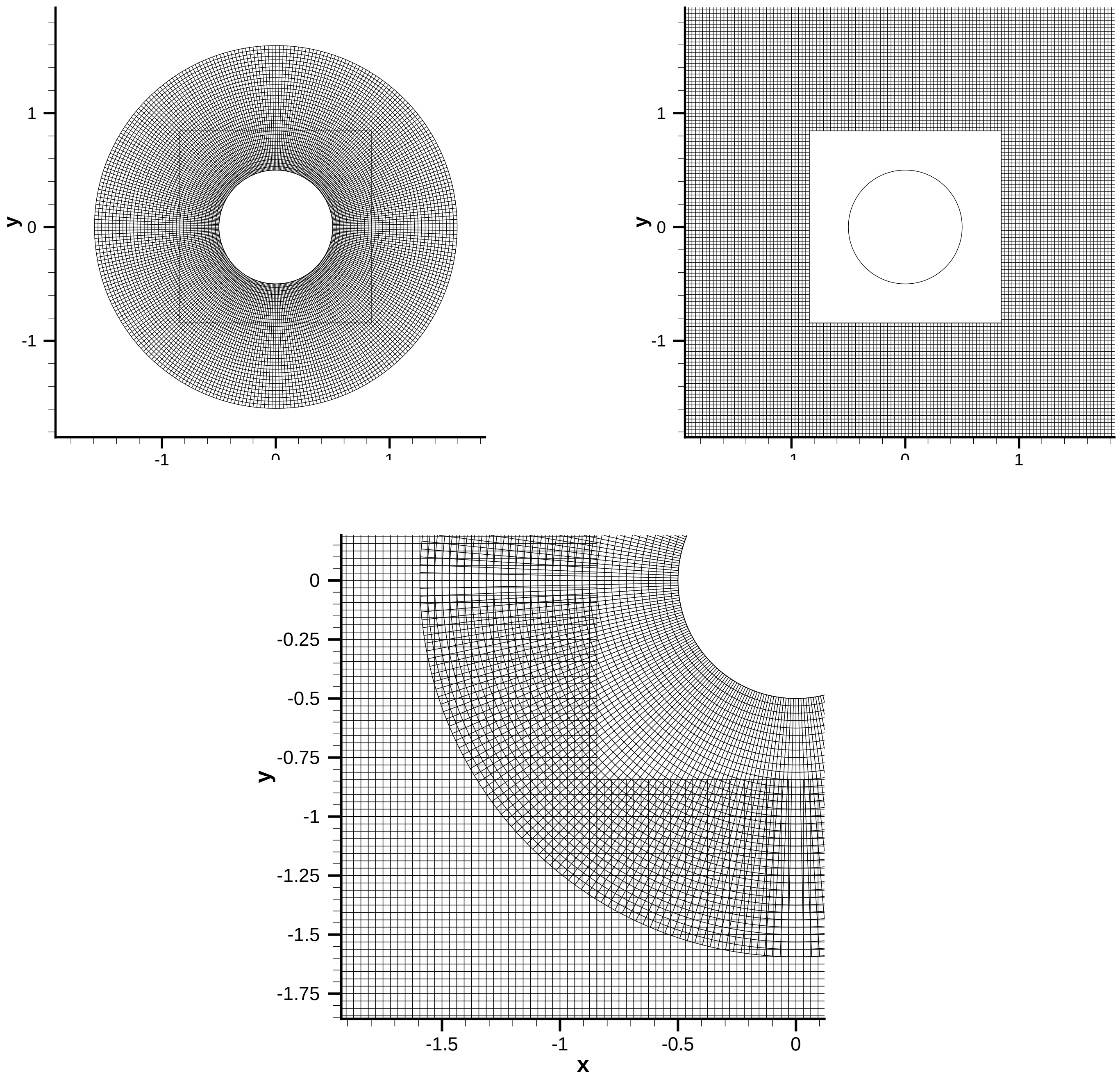
Wavenumber space:

$$u(k) = [\cdots]\,\text{exp}[-\frac{R_{\text{stencil}}}{\Delta x} D(k\Delta x)]$$

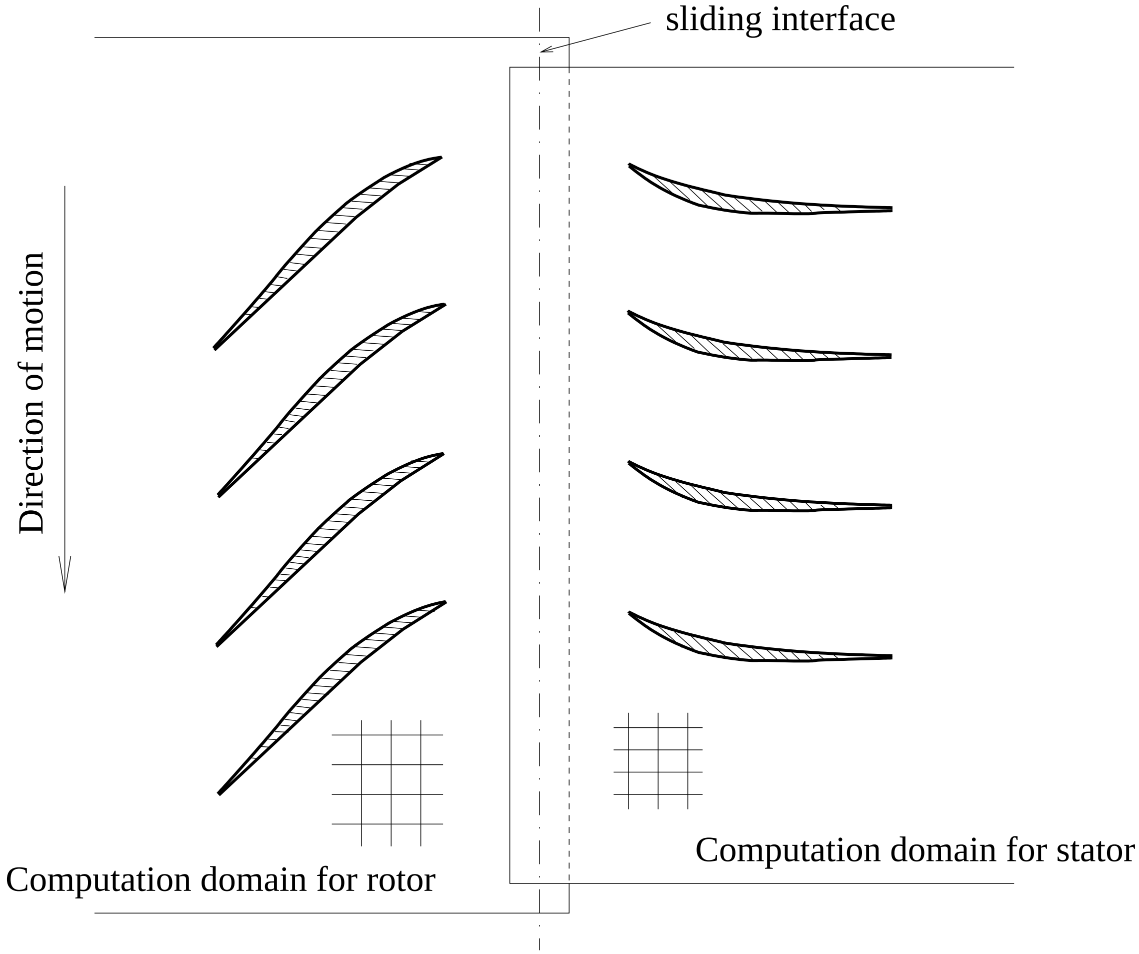
$$D(k\Delta x) = d_0 + 2\sum_{\ell=1}^{3} d_\ell \cos(\ell k \Delta x)$$


DRP artificial damping curve
DRP I, Tam&Webb, JCP, 1993
DRP II
D(kΔx)
kΔx
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3

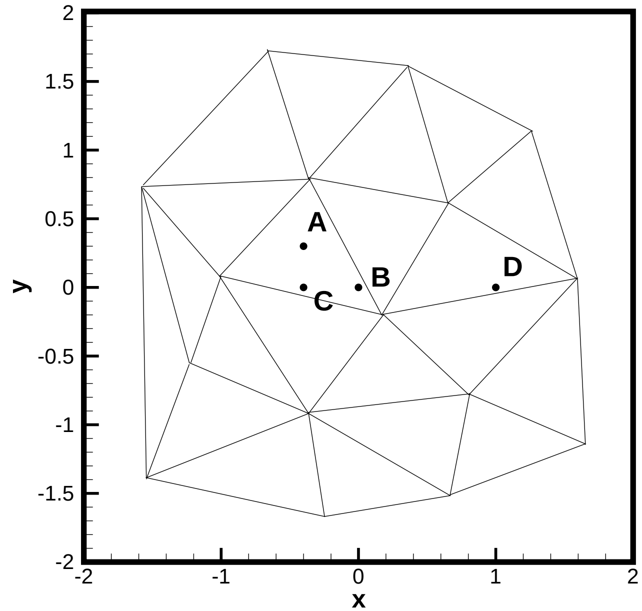
Complex geometries - Overset grid



Sliding interfaces



Fourier analysis of interpolation errors



$$f(x_0, y_0) = \sum_{j=1}^{N} S_j f(x_j, y_j)$$

Assume $$f(x, y) = e^{i(\alpha x + \beta y + \phi(\alpha,\beta))}$$

Error: $$E^2 = \left| e^{i(\alpha x_0 + \beta y_0 + \phi(\alpha,\beta))} - \sum_{j=1}^{N} S_j e^{i(\alpha x_j + \beta y_j + \phi(\alpha,\beta))} \right|^2 = \left| 1 - \sum_{j=1}^{N} S_j e^{i(\alpha(x_j - x_0) + \beta(y_j - y_0))} \right|^2$$

Interpolation schemes

1. Polynomial interpolation

$$f(x,y) = \sum_{j=1}^{N} a_j \phi(x,y), \qquad \phi(x,y) = (1, x, x^2, x^3) \otimes (1, y, y^2, y^3)$$

choose a_j such that

$$\sum_{j=1}^{N} a_j \phi(x_i, y_i) = f_i$$

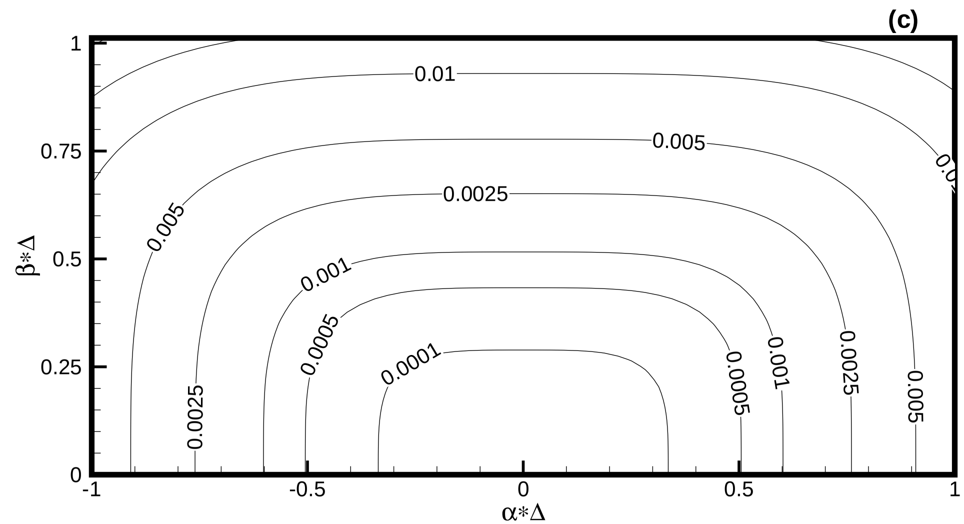
2. Lagrange interpolation (regular grids)

$$f(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} [\prod_{\ell=0,\ell\neq i}^{3} \frac{(x - x_\ell)}{(x_i - x_\ell)} \prod_{k=0,k\neq j}^{3} \frac{(y - y_k)}{(y_j - y_k)}] f(x_i, y_j)$$

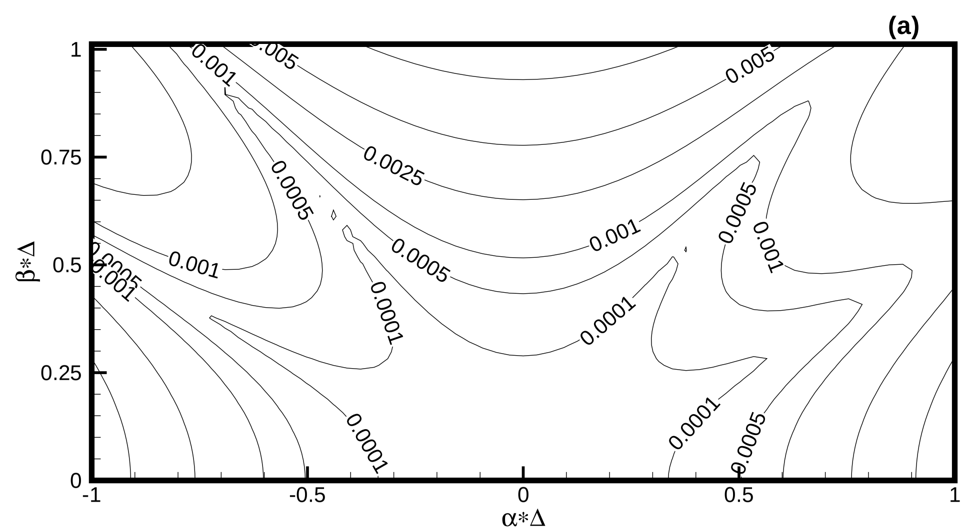
3. Optimized interpolation

$$\int \int_{-\kappa}^{\kappa} \left| 1 - \sum_{j=1}^{N} S_j e^{i[\alpha\Delta(\frac{x_j - x_0}{\Delta}) + \beta\Delta(\frac{y_j - y_0}{\Delta})]} \right|^2 d(\alpha\Delta) d(\beta\Delta) = \text{MINIMUM}$$

Interpolation errors

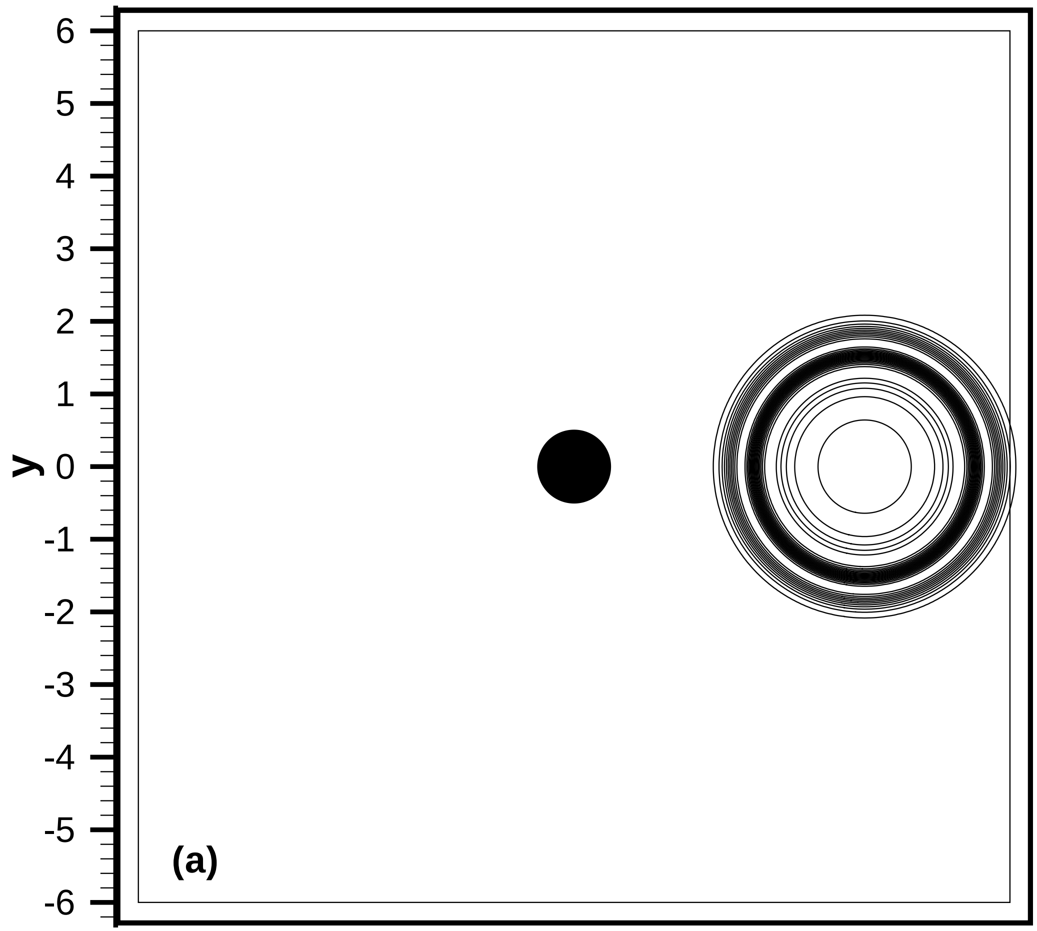


Lagrange interpolation

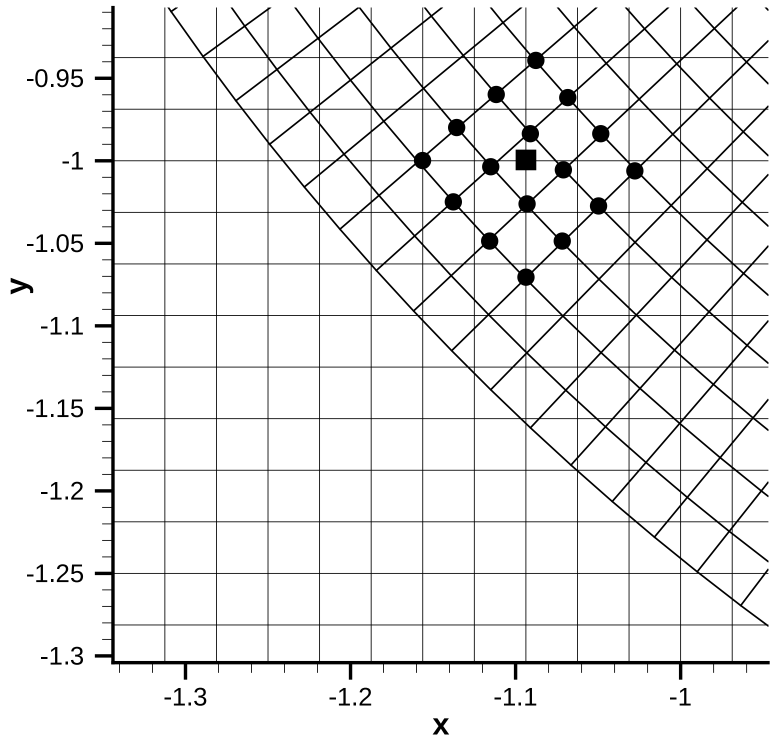


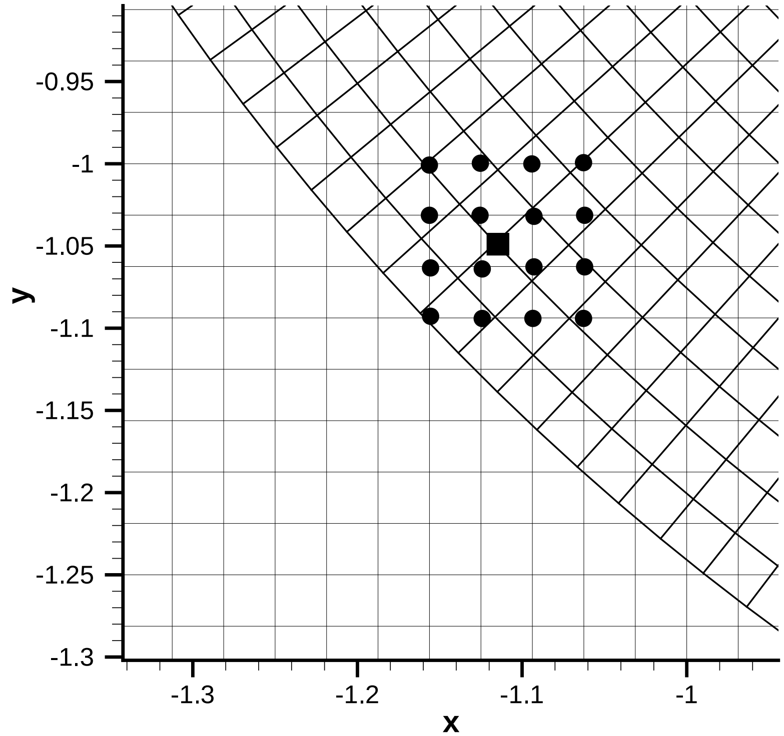
Optimized interpolation

Example: cylinder scattering

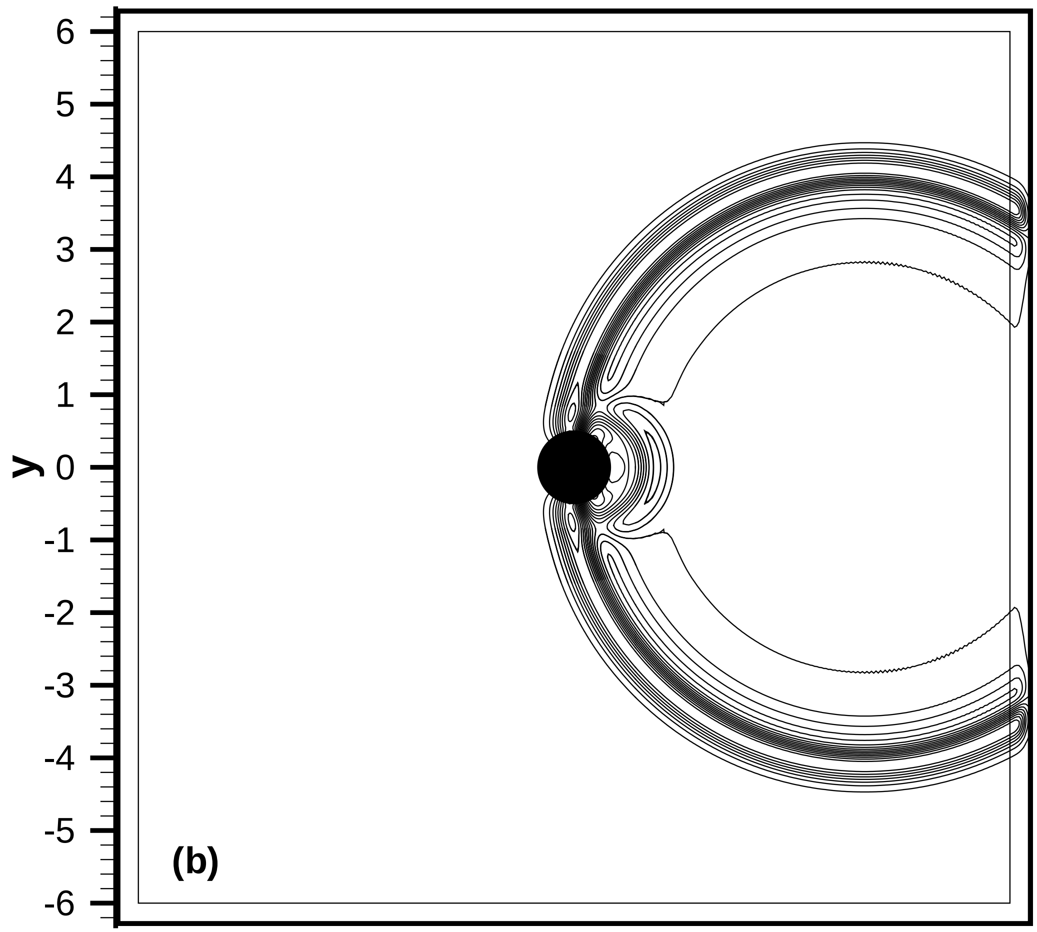


Interpolation between overset grids

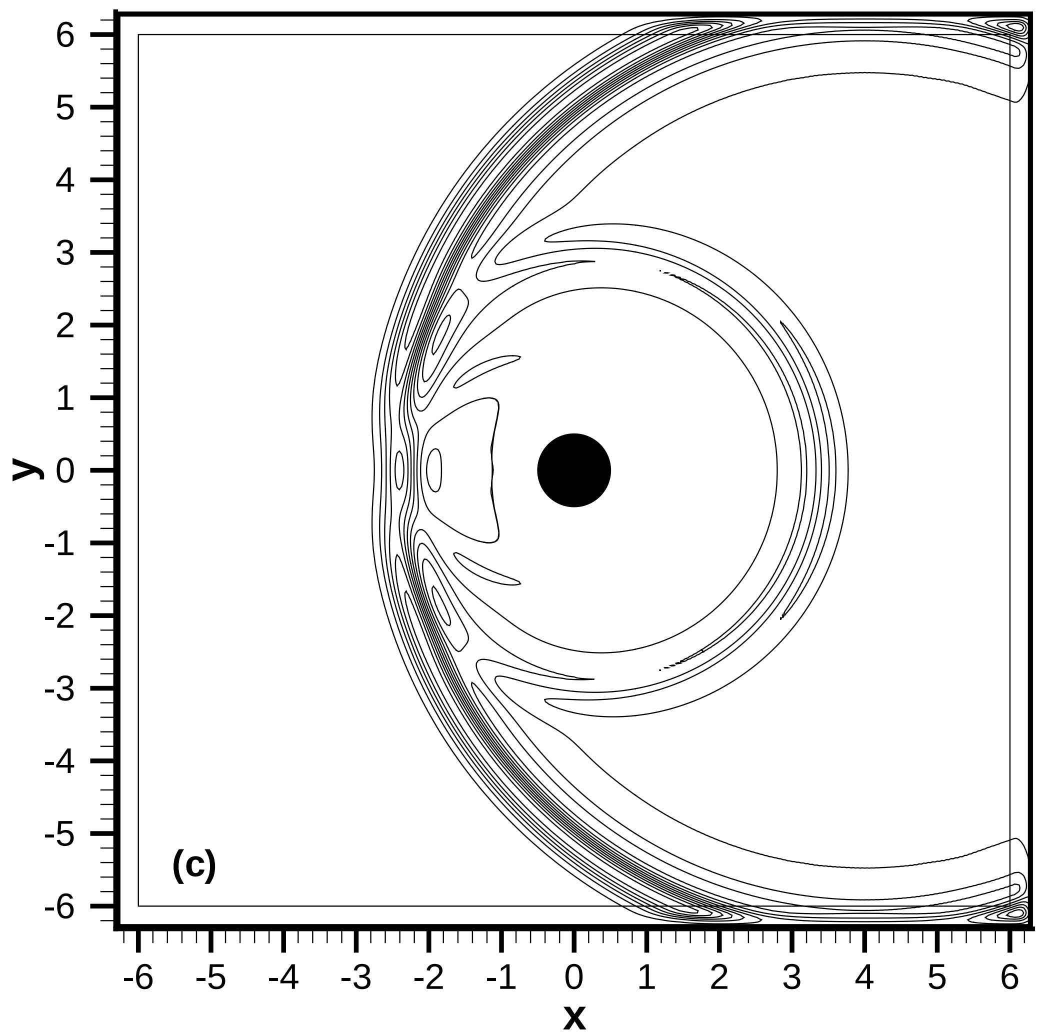




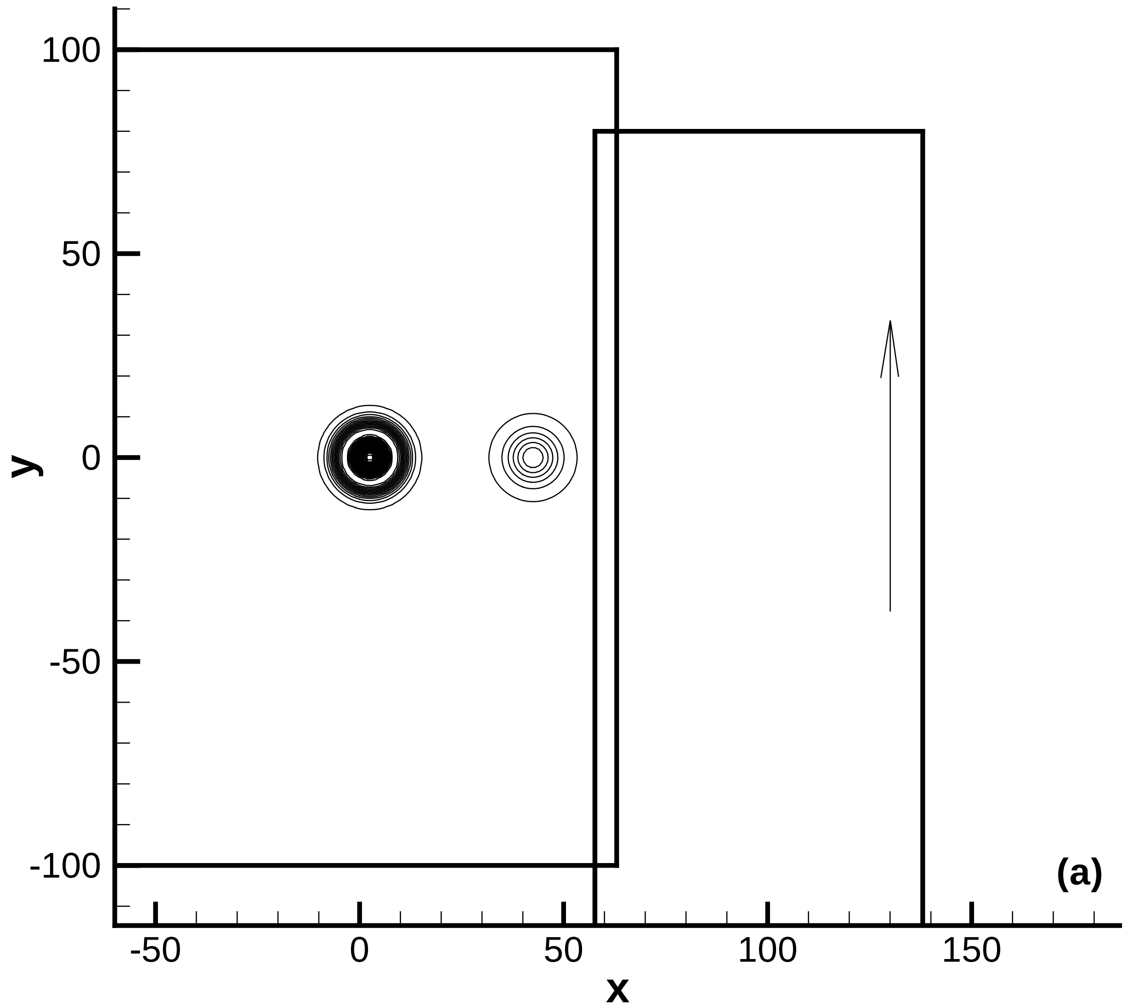
Pressure Contours



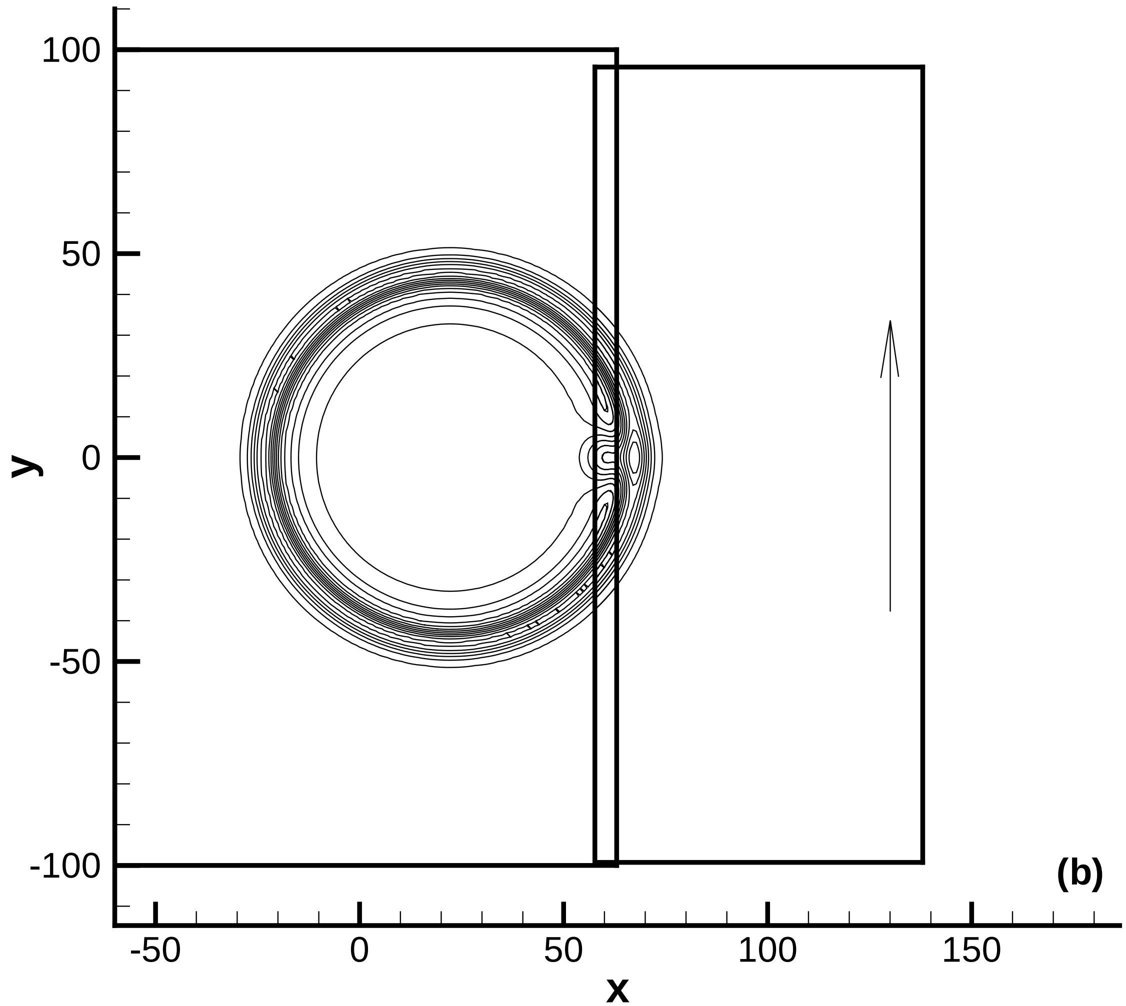
Pressure Contours



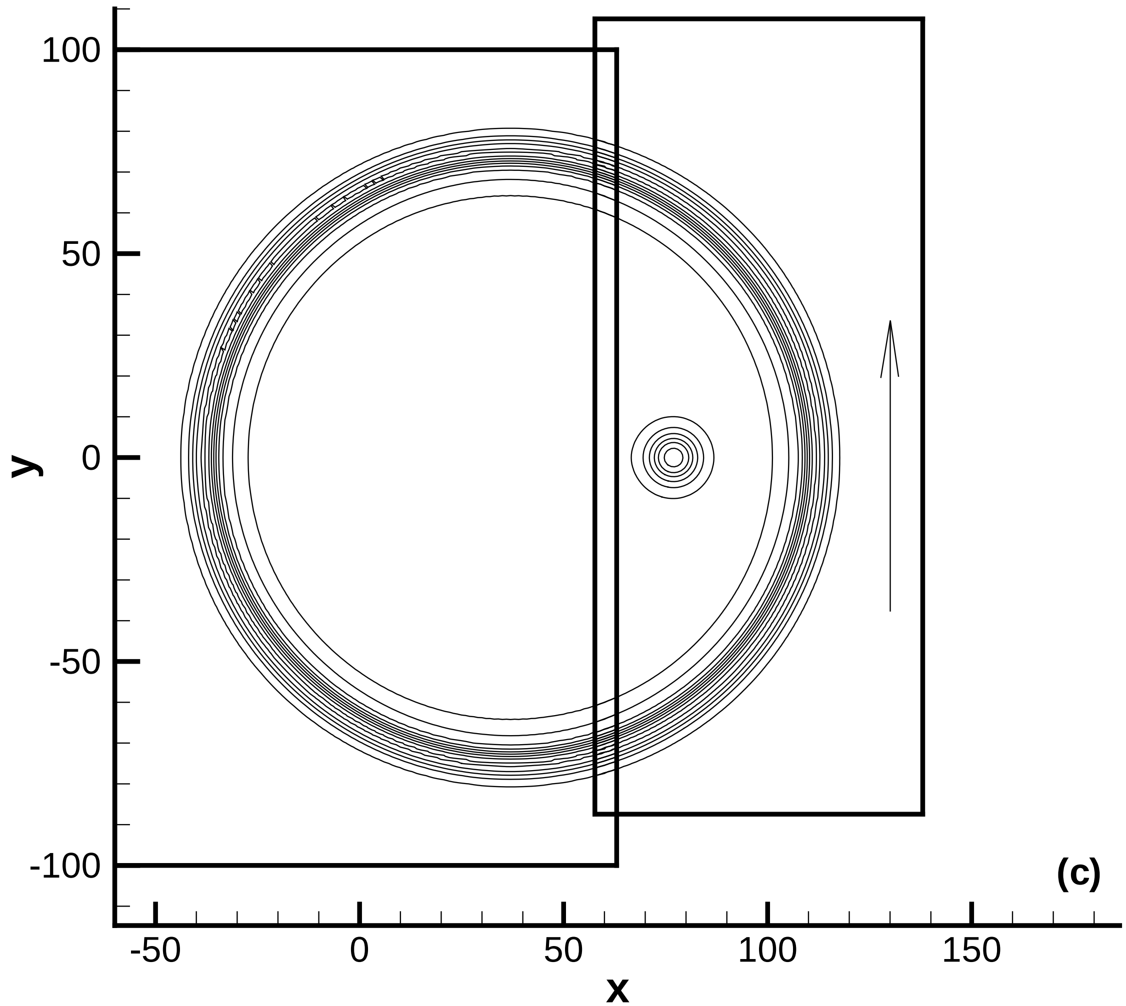
Example: sliding interface



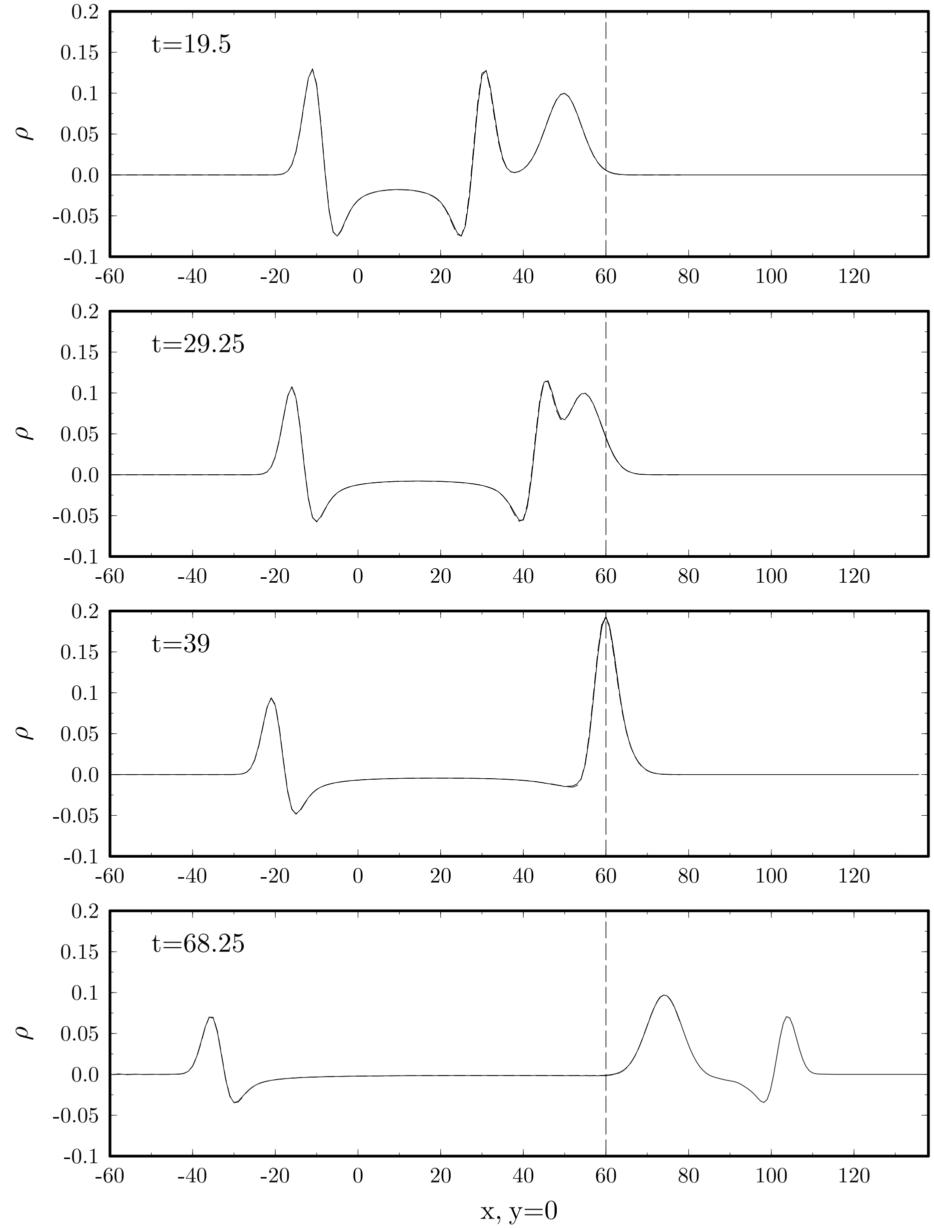
Example: sliding interface



Example: sliding interface



Density profile



Part II

- ⧫ A comparison of finite difference and finite element methods
- ⧫ Discontinuous Galerkin Finite Element Method
- ⧫ Boundary Element Method

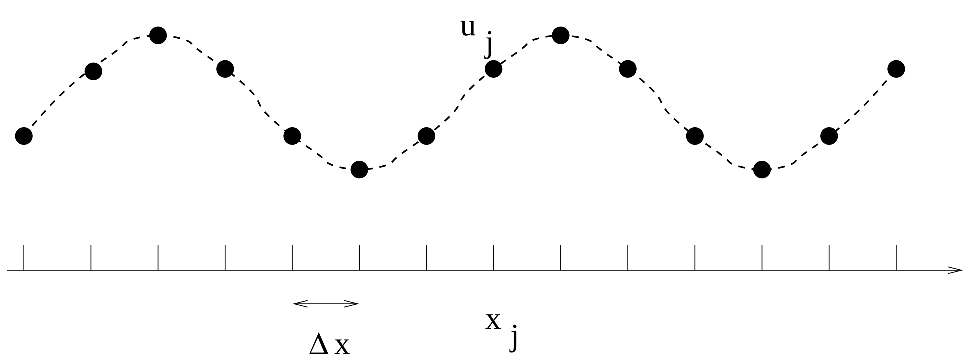
Central Difference Schemes

- A model equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

- Semi-discrete equation (4th-order):

$$\frac{du_j}{dt} + \frac{u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}}{12\Delta x} = 0$$



Central Difference Schemes
— Fourier Analysis

- Let

$$u_j = \widehat{u}(t) e^{ikx_j}$$

- Semi-discrete equation:

$$\frac{d\widehat{u}}{dt} + ik^*\widehat{u} = 0$$

Numerical wavenumber (4th-oder):

$$k^* = -\frac{i(e^{-2ik\Delta x} - 8e^{-ik\Delta x} + 8e^{ik\Delta x} - e^{2ik\Delta x})}{12\Delta x}$$

$$= \frac{8\sin(k\Delta x) - \sin(2k\Delta x)}{6\Delta x}$$

- Numerical phase speed (assume exact time integration):

$$c^* = \frac{k^*}{k} = \frac{8\sin(k\Delta x) - \sin(2k\Delta x)}{6k\Delta x} = 1 + C_0(k\Delta x)^4 + \cdots$$

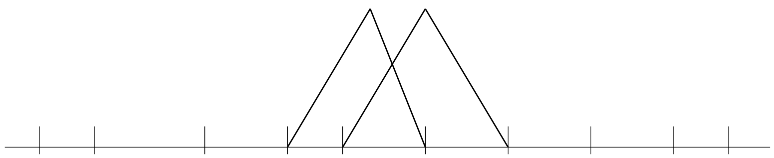
Finite Element Methods

- A model equation:

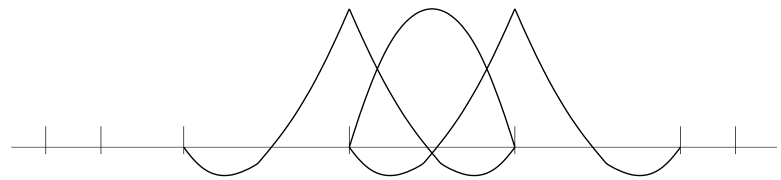
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

- Expansion:

$$u(x,t) = \sum_j c_j(t)\phi_j(x), \quad \phi_j(x): \text{ basis functions}$$



Linear Elements



Ouadratic Elements

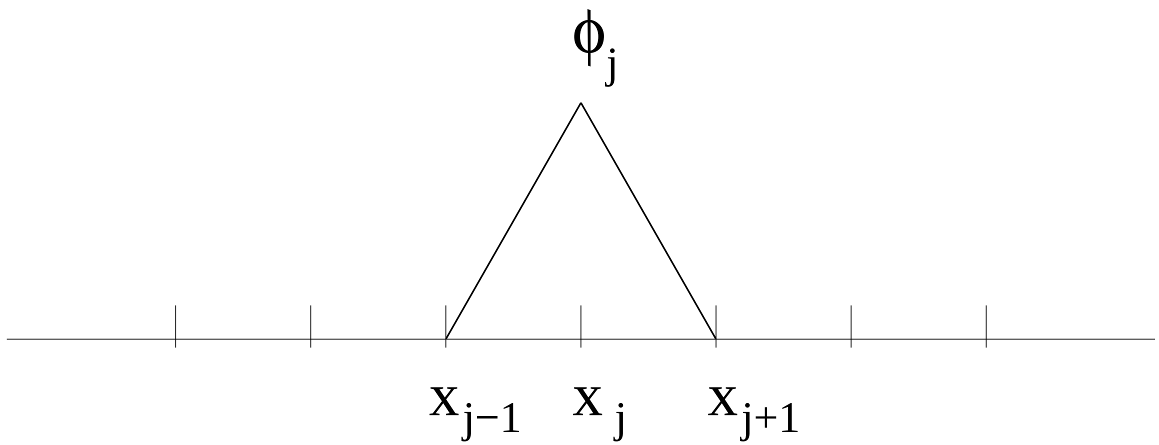
- Galerkin formulation:

$$\int_\Omega \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right)\phi_n dx = 0$$

- Semi-discrete equation (linear elements):

$$\sum_j \frac{dc_j}{dt}\int_\Omega \phi_j\phi_n dx + \sum_j \int_\Omega \frac{d\phi_j}{dx}\phi_n dx = 0$$

Example: Linear elements with uniform grids



$$\int_\Omega \phi_j \phi_n dx = \Delta x \begin{cases} \frac{1}{6} & n = j-1 \\ \frac{2}{3} & n = j \\ \frac{1}{6} & n = j+1 \end{cases}$$

$$\int_\Omega \frac{d\phi_j}{dx} \phi_n dx = \begin{cases} \frac{1}{2} & n = j-1 \\ 0 & n = j \\ -\frac{1}{2} & n = j+1 \end{cases}$$

$$\sum_j \frac{dc_j}{dt} \int_\Omega \phi_j \phi_n dx + \sum_j \int_\Omega \frac{d\phi_j}{dx} \phi_n dx = 0$$

$$\Longrightarrow \frac{1}{6}\frac{dc_{j-1}}{dt} + \frac{4}{6}\frac{dc_j}{dt} + \frac{1}{6}\frac{dc_{j+1}}{dt} + \frac{c_{j+1} - c_{j-1}}{2\Delta x} = 0$$

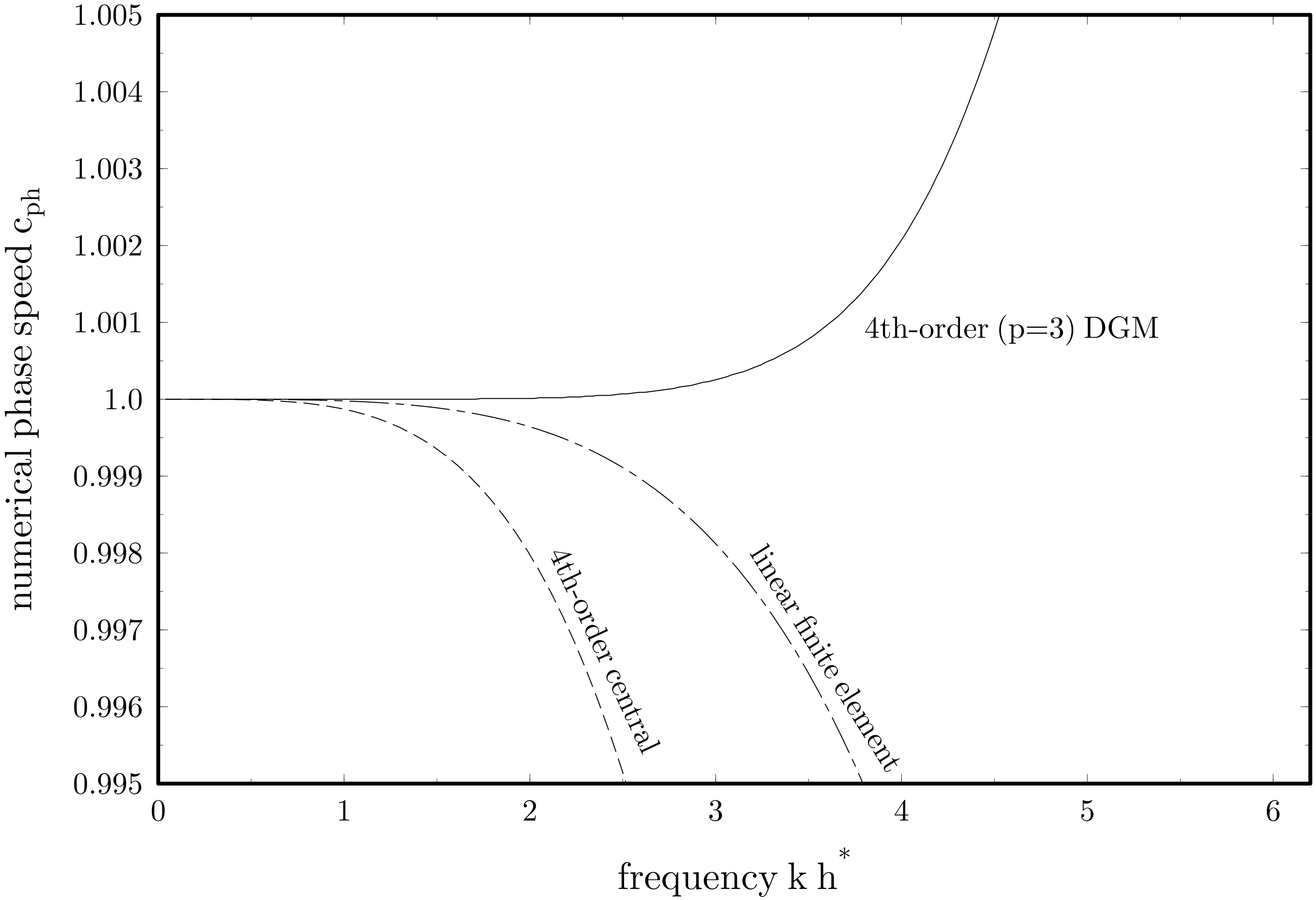
- Equivalent to a 4th-order compact scheme.
- Semi-discrete equation for $c_j = \hat{c}(t)e^{ikx_j}$:

$$\frac{d\hat{c}}{dt} + ik^*\hat{c} = 0$$

- Numerical phase speed:

$$c^* = \frac{k^*}{k} = \frac{3\sin(k\Delta x)}{k\Delta x[2 + \cos(k\Delta x)]} = 1 + C_0(k\Delta x)^4 + \cdots$$

Comparison of numerical phase speeds

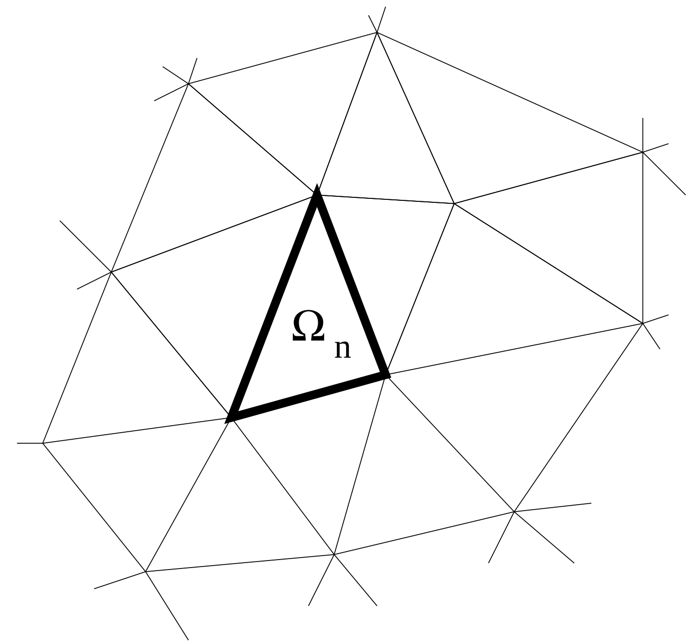


Discontinuous Galerkin Formulation

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \vec{\mathbf{f}}(\mathbf{u}) = 0$$

- Approximation in element Ω_n by polynomial expansion:

$$\mathbf{u}^n(\mathbf{x}, t) = \sum_{\ell=0}^{N} \mathbf{c}_\ell^n(t)\phi_\ell(\mathbf{x})$$



- Weak formulation:

$$\int_{\Omega_n} \left\{ \frac{\partial \mathbf{u}^n}{\partial t} + \nabla \cdot \vec{\mathbf{f}}(\mathbf{u}^n) \right\} \phi_\ell(\mathbf{x}) d\Omega = 0$$

- Integration by parts:

$$\int_{\Omega_n} \frac{\partial \mathbf{u}^n}{\partial t} \phi_\ell(\mathbf{x}) d\Omega + \int_{\partial\Omega_n} [\vec{\mathbf{f}}(\mathbf{u}^n) \cdot \mathbf{n}] \phi_\ell ds - \int_{\Omega_n} \vec{\mathbf{f}}(\mathbf{u}^n) \cdot \nabla \phi_\ell d\Omega = 0$$

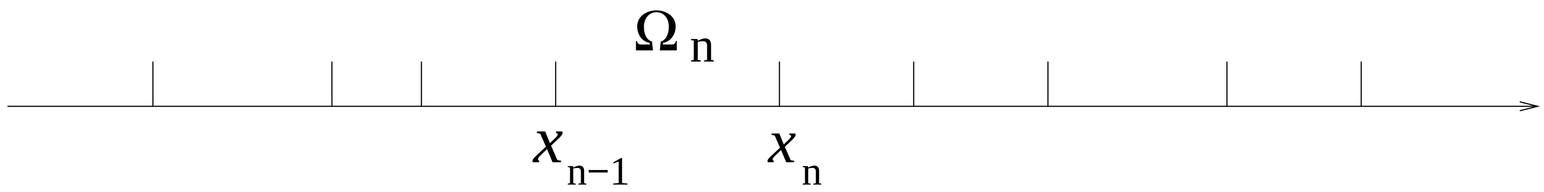
$$\Uparrow$$

inter – element communication

Advantages of Discontinuous Galerkin Method

- Robust for using high order basis polynomials
- Uses unstructured meshes for complex geometries
- Highly compact , good for parallel implementation
- Low dispersion and dissipation errors

One Space Dimension



$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0, \quad \mathbf{f}(\mathbf{u}) = \mathbf{A}\mathbf{u}, \quad \text{eig}(\mathbf{A}) = \{a_j, \mathbf{e}_j\}$$

- Expansion in $[x_{n-1}, x_n]$:

$$\mathbf{u}^n(x,t) = \sum_{\ell=0}^{p} \mathbf{c}_\ell^n(t)\phi_\ell(x), \quad p = \text{highest order}$$

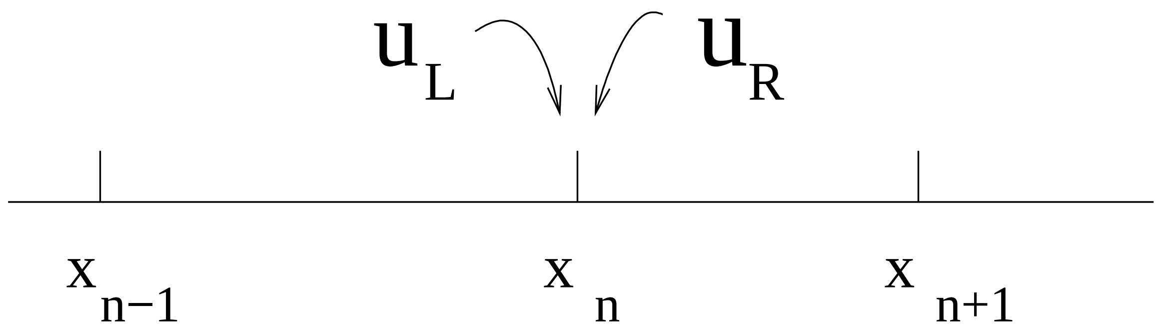
- Weak formulation:

$$\int_{x_{n-1}}^{x_n} \left\{ \frac{\partial \mathbf{u}^n}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u}^n)}{\partial x} \right\} \phi_\ell(x) dx = 0$$

- Integration by parts:

$$\int_{x_{n-1}}^{x_n} \frac{\partial \mathbf{u}^n}{\partial t} \phi_\ell(x) dx + [\mathbf{f}(\mathbf{u}^n)\phi_\ell(x)]_{x_{n-1}}^{x_n} - \int_{x_{n-1}}^{x_n} \mathbf{f}(\mathbf{u}^n) \frac{\partial \phi_\ell}{dx} dx = 0$$

Flux Formulas



- Characteristics-splitting flux formula:

$$\mathbf{f}(\mathbf{u}^n) = \mathbf{A}\mathbf{u} = \mathbf{A}_+\mathbf{u} + \mathbf{A}_-\mathbf{u} = \mathbf{A}_+\mathbf{u}_L + \mathbf{A}_-\mathbf{u}_R$$

- Lax-Friedrich flux formula:

$$\mathbf{f}(\mathbf{u}^n) = \frac{1}{2}[\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)] - \frac{1}{2}|a|_{max}(\mathbf{u}_R - \mathbf{u}_L), \quad a = eig(\mathbf{A})$$

- Combined notation:

$$\mathbf{f}(\mathbf{u}^n) = \mathbf{A}_L\mathbf{u}_L + \mathbf{A}_R\mathbf{u}_R$$

Semi-discrete Equation

$$\int_{x_{n-1}}^{x_n} \frac{\partial \mathbf{u}^n}{\partial t} \phi_\ell(x) dx + \left[\mathbf{A}_L \mathbf{u}^n + \mathbf{A}_R \mathbf{u}^{n+1}\right] \phi_\ell(x_n) - \left[\mathbf{A}_L \mathbf{u}^{n-1} + \mathbf{A}_R \mathbf{u}^n\right] \phi_\ell(x_{n-1}) - \int_{x_{n-1}}^{x_n} \mathbf{A}\mathbf{u}^n \frac{\partial \phi_\ell}{dx} dx = 0$$

$$\text{where } \mathbf{u}^n(t) = \sum_{\ell=0}^{p} \mathbf{c}_\ell^n(t) \phi_\ell(x)$$

- Semi-discrete equation for the expansion coefficients:

$$\text{Define } \mathbf{C}^n(t) = [\mathbf{c}_0^n(t), \mathbf{c}_1^n(t), \cdots\cdots, \mathbf{c}_p^n(t)]^T$$

$$\mathbf{Q} \frac{\partial \mathbf{C}_j^n}{\partial t} + (1+\gamma)\mathbf{B}_{-1}\mathbf{C}_j^{n-1} + \mathbf{B}_0 \mathbf{C}_j^n + (1-\gamma)\mathbf{B}_1 \mathbf{C}_j^{n+1} = 0$$

- Flux parameter γ:

$$\gamma = \frac{|a_j|}{a_j} = \pm 1 \qquad \Longleftarrow \text{characteristics-splitting}$$

$$\gamma = \frac{|a_{max}|}{a_j} \qquad \Longleftarrow \text{Lax-Friedrich}$$

Disretization matrices

$$\mathbf{Q} = \{q_{\ell'\ell}\}_{(p+1)\times(p+1)} \qquad q_{\ell'\ell} = \int_{-1}^{1} \phi_\ell(\xi)\phi_{\ell'}(\xi)d\xi$$

$$\mathbf{B}_1 = \{\phi_{\ell'}(1)\phi_\ell(-1)\}_{(p+1)\times(p+1)}$$

$$\mathbf{B}_{-1} = -\{\phi_{\ell'}(-1)\phi_\ell(1)\}_{(p+1)\times(p+1)}$$

Fourier Analysis

$$\mathbf{Q}\frac{\partial \mathbf{C}^n}{\partial t} + (1+\gamma)\mathbf{B}_{-1}\mathbf{C}^{n-1} + \mathbf{B}_0\mathbf{C}^n + (1-\gamma)\mathbf{B}_1\mathbf{C}^{n+1} = 0$$

- Wave form:

$$\mathbf{C}^n(t) = e^{-i\omega t}e^{ikx_n}\tilde{\mathbf{C}}, \qquad x_n = nh$$

- Eigenvalue problem:

$$-i\omega\mathbf{Q}\tilde{\mathbf{C}} + (1+\gamma)e^{-ikh}\mathbf{B}_{-1}\tilde{\mathbf{C}} + \mathbf{B}_0\tilde{\mathbf{C}} + (1-\gamma)e^{ikh}\mathbf{B}_1\tilde{\mathbf{C}} = 0$$

$$|-i\omega\mathbf{Q} + (1+\gamma)e^{-ikh}\mathbf{B}_{-1} + \mathbf{B}_0 + (1-\gamma)e^{ikh}\mathbf{B}_1| = 0$$

Determinant, $\gamma = 1$(exact characteristics-splitting)

$$\left|-i\omega \mathbf{Q} + 2e^{-ikh}\mathbf{B}_{-1} + \mathbf{B}_0\right| = 0$$

- $p = 1$:

$$\left[1 - \frac{2}{3}(i\Omega) + \frac{1}{6}(i\Omega)^2\right] - \left[1 + \frac{1}{3}i\Omega\right] e^{-iK} = 0, \qquad \Omega = \frac{\omega h}{a_j}, K = kh$$

- $p = 2$:

$$\left[1 - \frac{3}{5}(i\Omega) + \frac{3}{20}(i\Omega)^2 - \frac{1}{60}(i\Omega)^3\right] - \left[1 + \frac{2}{5}(i\Omega) + \frac{1}{20}(i\Omega)^2\right] e^{-iK} = 0$$

- $p = 3$:

$$\left[1 - \frac{4}{7}(i\Omega) + \frac{1}{7}(i\Omega)^2 - \frac{2}{105}(i\Omega)^3 + \frac{1}{840}(i\Omega)^4\right] - \left[1 + \frac{3}{7}(i\Omega) + \frac{1}{24}(i\Omega)^2 + \frac{1}{200}(i\Omega)^3\right] e^{-iK} = 0$$

- • •

$$f(i\Omega) - g(i\Omega)e^{-iK} = 0$$

$$\Longrightarrow e^{iK} = \frac{g(i\Omega)}{f(i\Omega)} = e^{i\Omega} + O\left((i\Omega)^{2p+2}\right) \Longleftarrow \text{pade approximation of } e^{i\Omega}$$

$$\Longrightarrow K = \Omega + O\left(\Omega^{2p+2}\right)$$

Determinant, $\gamma \neq 1$

$$|-i\omega \mathbf{Q} + (1+\gamma)e^{-ikh}\mathbf{B}_{-1} + \mathbf{B}_0 + (1-\gamma)e^{ikh}\mathbf{B}_1| = 0$$

- Determinant:

$$f(i\Omega) + (1-\gamma)g_1(i\Omega)e^{iK} + (1+\gamma)g_2(i\Omega)e^{-iK} = 0$$

(Quadratic equation for e^{iK}, having two roots)

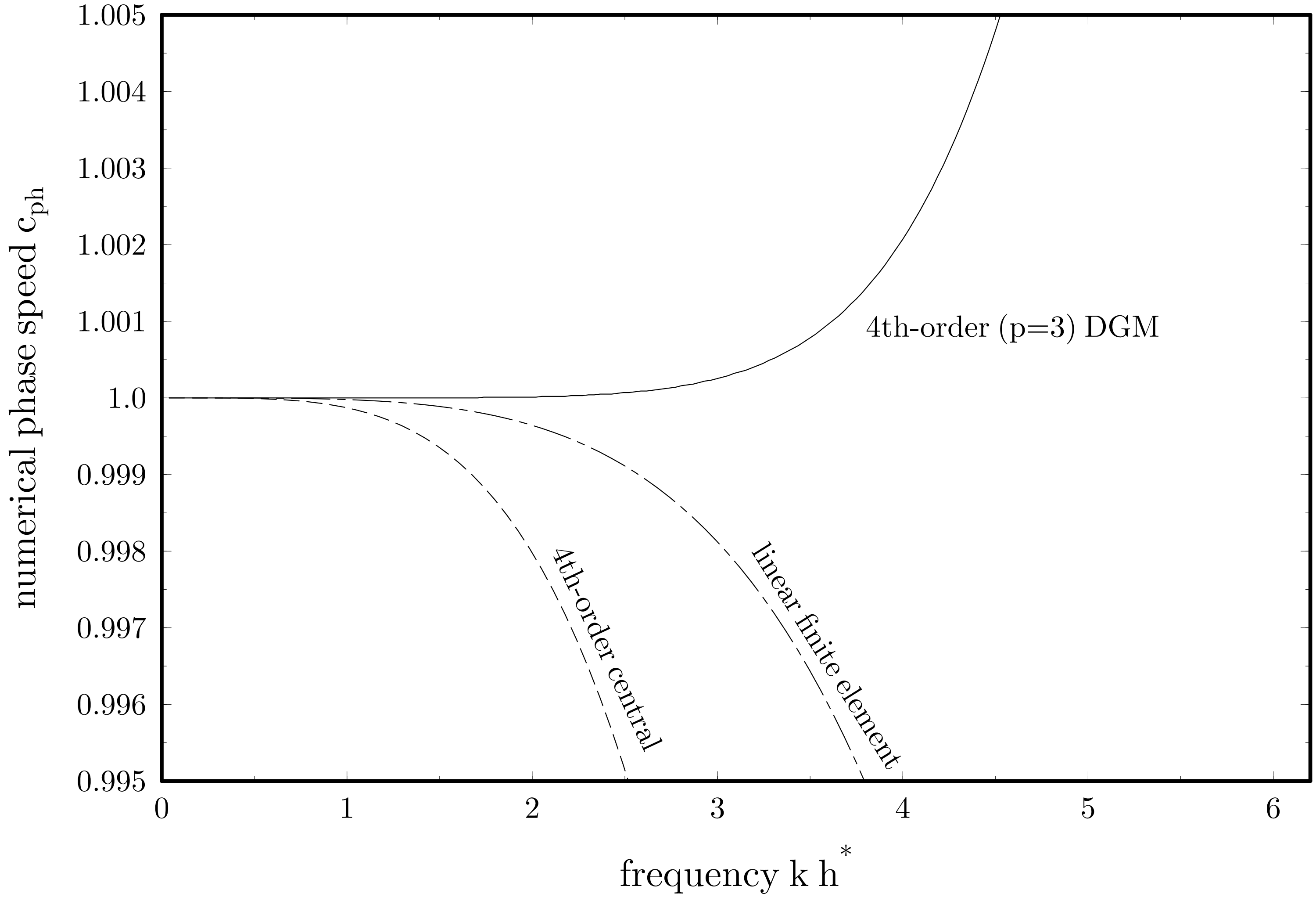
- Physical mode:

$$e^{iK} = e^{i\Omega} + C_1(i\Omega)^{2p+2} + ...$$

- Non-physical spurious mode:

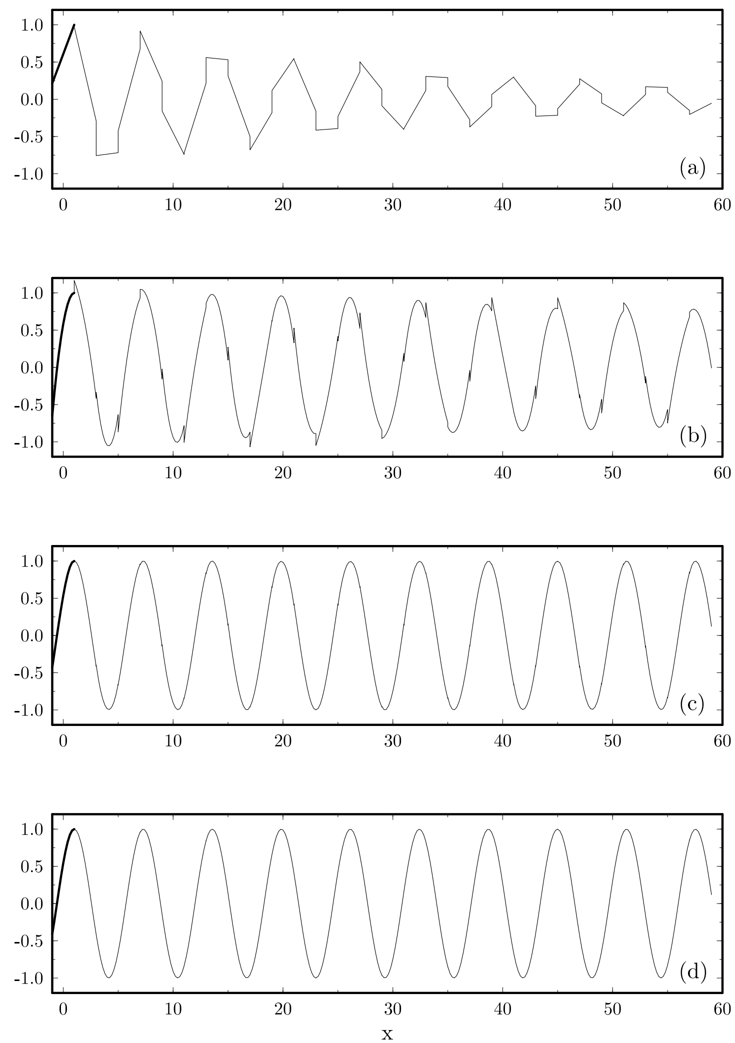
$$e^{iK} = D_0\frac{g(-i\Omega)}{g(i\Omega)}e^{-i\Omega} + D_1(i\Omega)^{2p+2} + ...$$

Comparison of numerical phase speeds



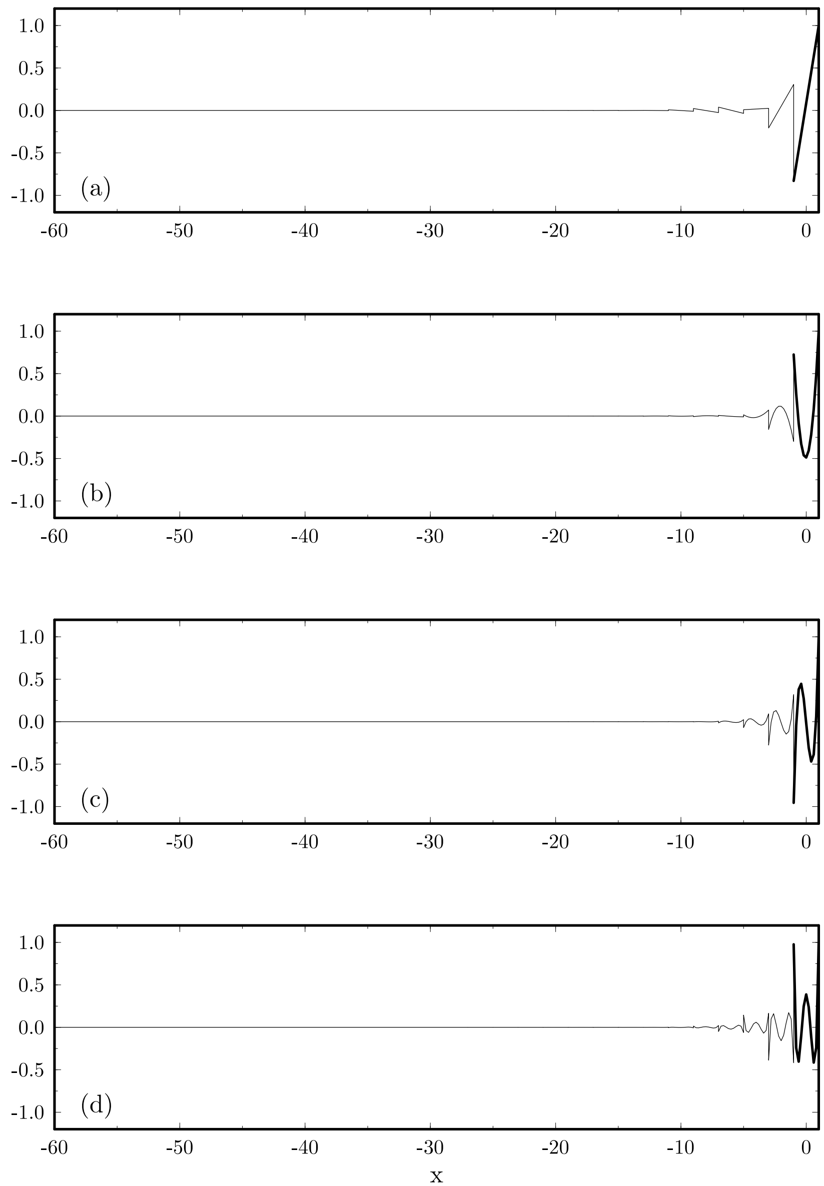
Physical Mode–Eigenfunctions

$(a)\ p = 1,\ \ (b)\ p = 2,\ \ (c)\ p = 3,\ \ (d)\ p = 4$



Spurious (Non-Physical) Mode–Eigenfunctions

$(a)\ p = 1, \quad (b)\ p = 2, \quad (c)\ p = 3, \quad (d)\ p = 4$

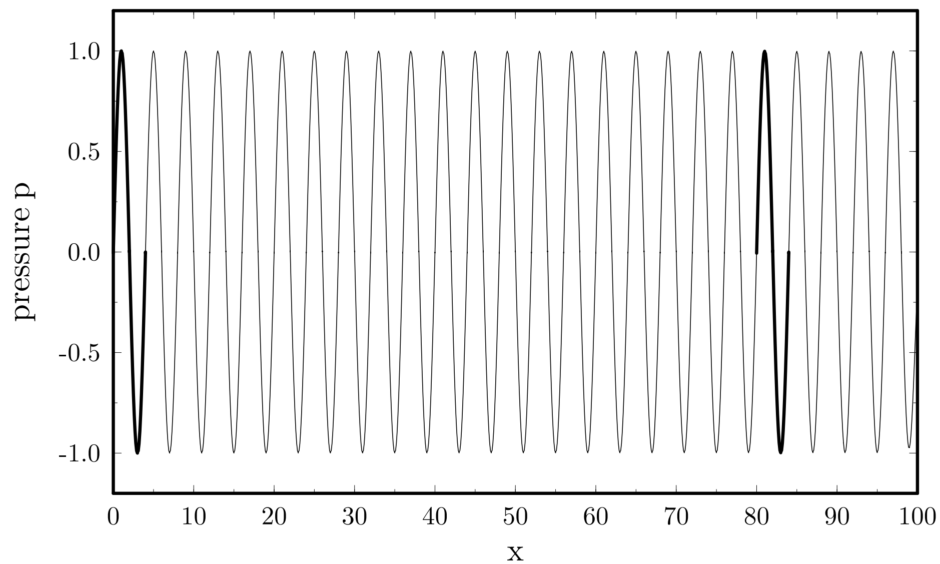


Numerical Example

$$\frac{\partial u}{\partial t} + M\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad (1)$$

$$\frac{\partial p}{\partial t} + M\frac{\partial p}{\partial x} + \frac{\partial u}{\partial x} = 0 \quad (2)$$

Boundary condition: incoming wave at $x = 0$, $\begin{bmatrix} u_{in} \\ p_{in} \end{bmatrix} = \sin[\omega_0(x-t)] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

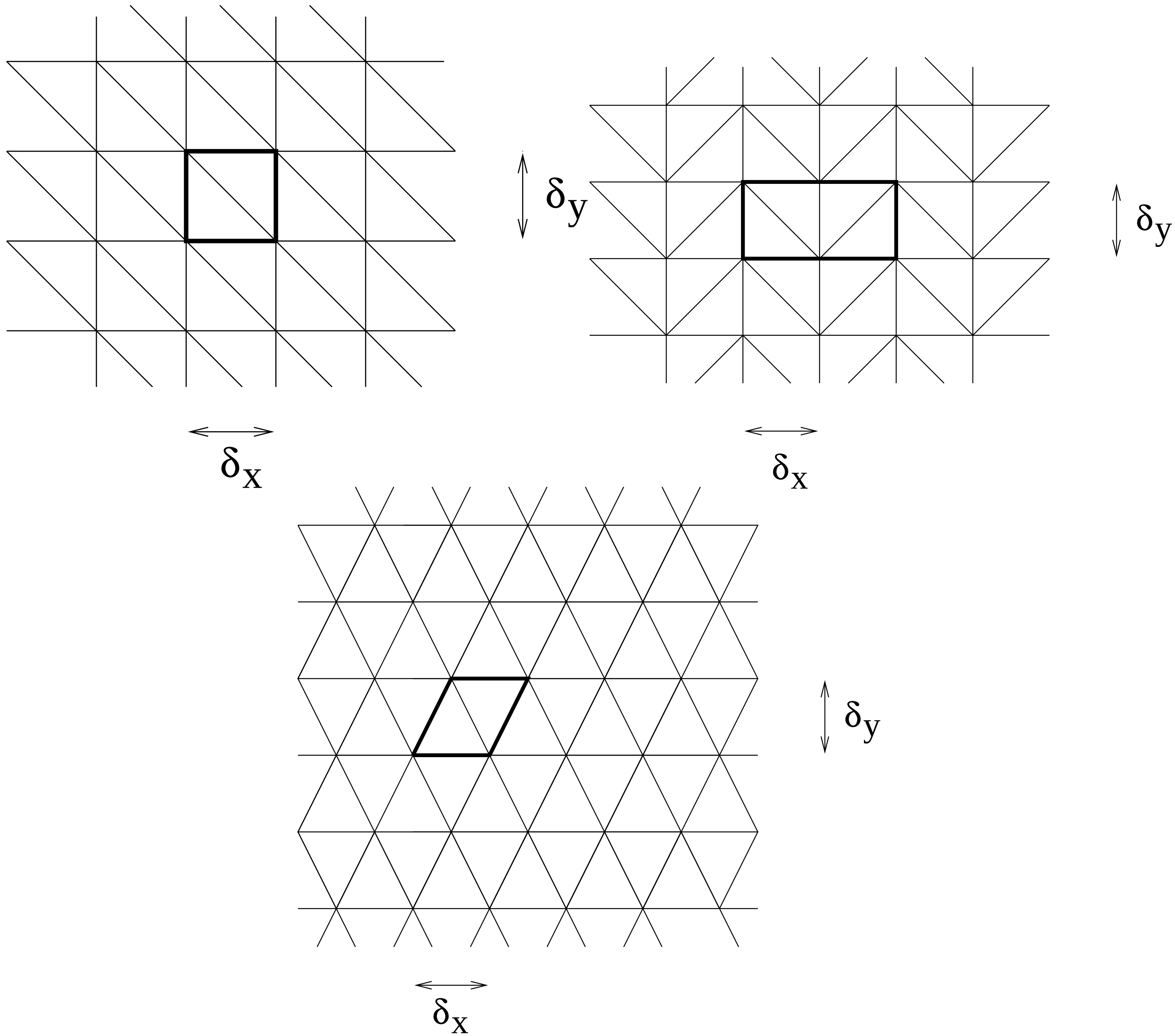


$$E = \sqrt{\int_0^{\lambda_0} |p_h(x,t) - p_h(x + 20\lambda_0, t)|^2 \, dx}$$

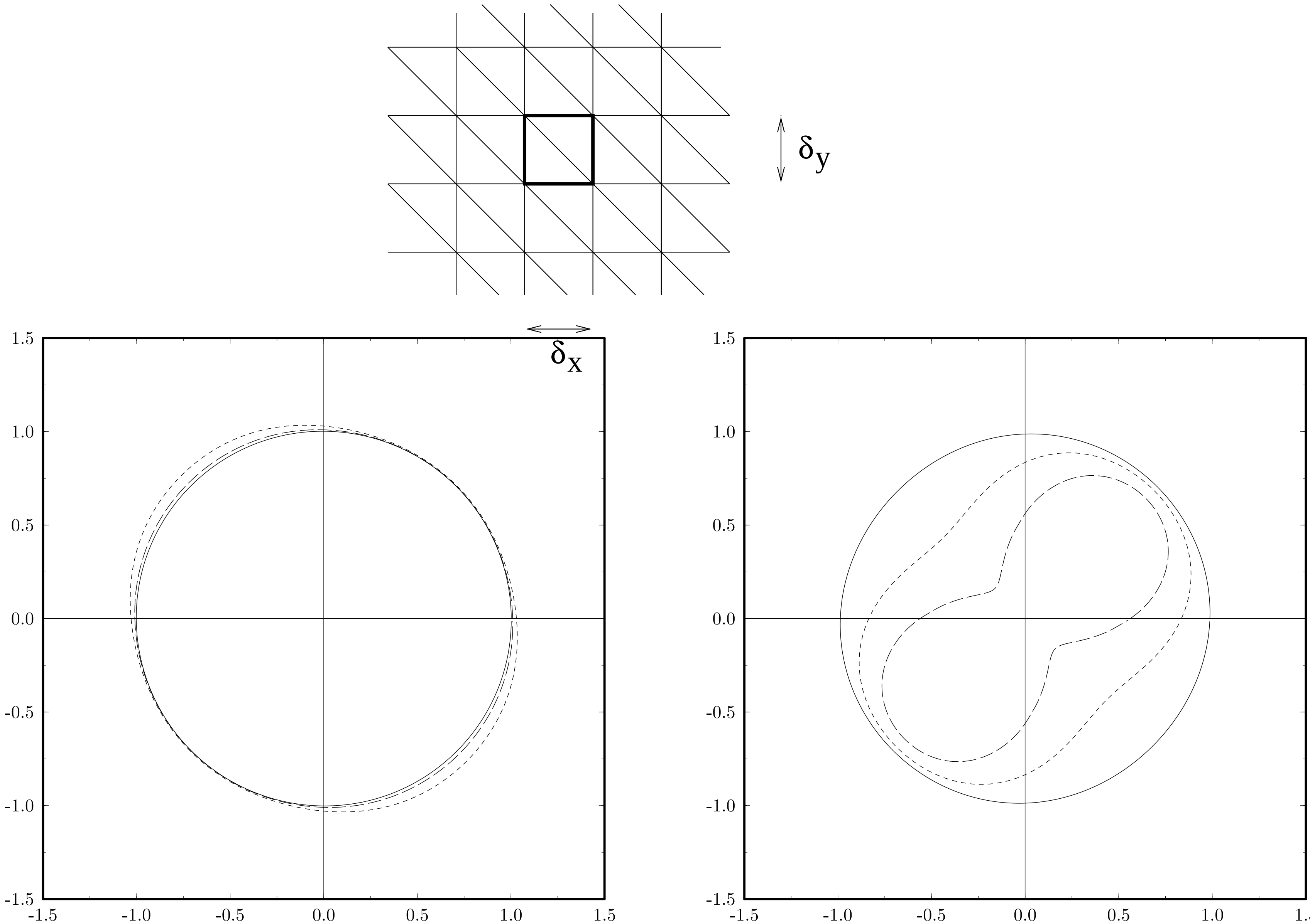
Mesh refinement results

		$\gamma = 1$		$\gamma = 0.5$	
p	h	Error E	order	Error E	order
	1	1.74054	-	1.79386	-
1	0.5	1.09166	0.6730	1.46813	0.2890
	0.25	0.197915	2.4635	0.344971	2.0894
	0.125	0.0261657	2.9191	0.0506057	2.7691
	1	0.27629	-	0.286715	-
2	0.5	0.010116	4.7714	0.00634575	5.4976
	0.25	0.000323692	4.9658	0.000172082	5.2053
	0.125	0.1016×10^{-4}	4.9923	0.5165×10^{-6}	5.0574
	1	0.00386958	-	0.00381781	-
3	0.5	0.3217×10^{-4}	6.9102	0.4912×10^{-4}	6.2801
	0.25	0.2552×10^{-6}	6.9780	0.4709×10^{-6}	6.7048
	0.125	0.2019×10^{-8}	6.9812	0.3964×10^{-8}	6.8919
	2	0.0126055	-	0.0238191	-
4	1	0.3002×10^{-4}	8.7137	0.3034×10^{-4}	9.6164
	0.5	0.6153×10^{-7}	8.9305	0.3858×10^{-7}	9.6192

Two dimensions

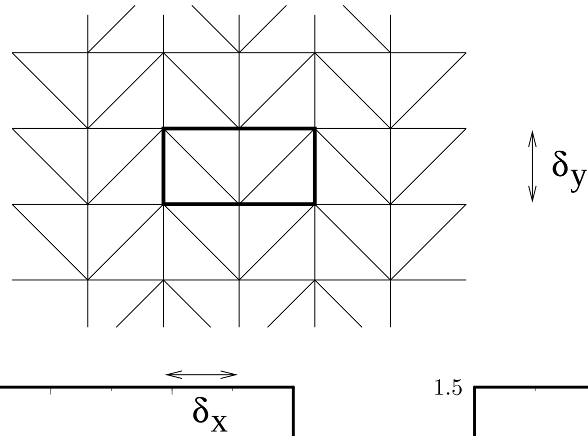


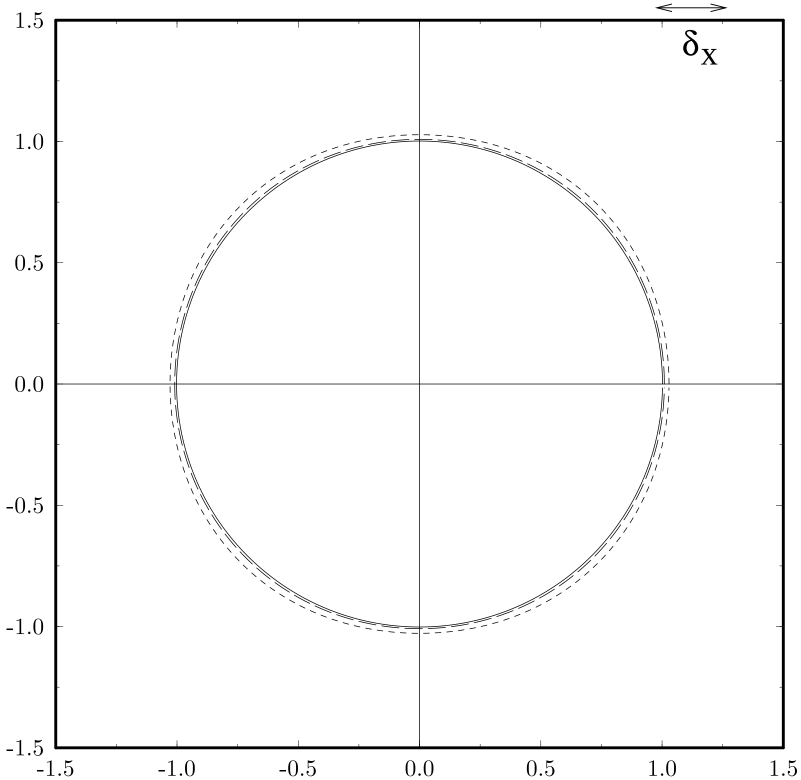
Two dimensions

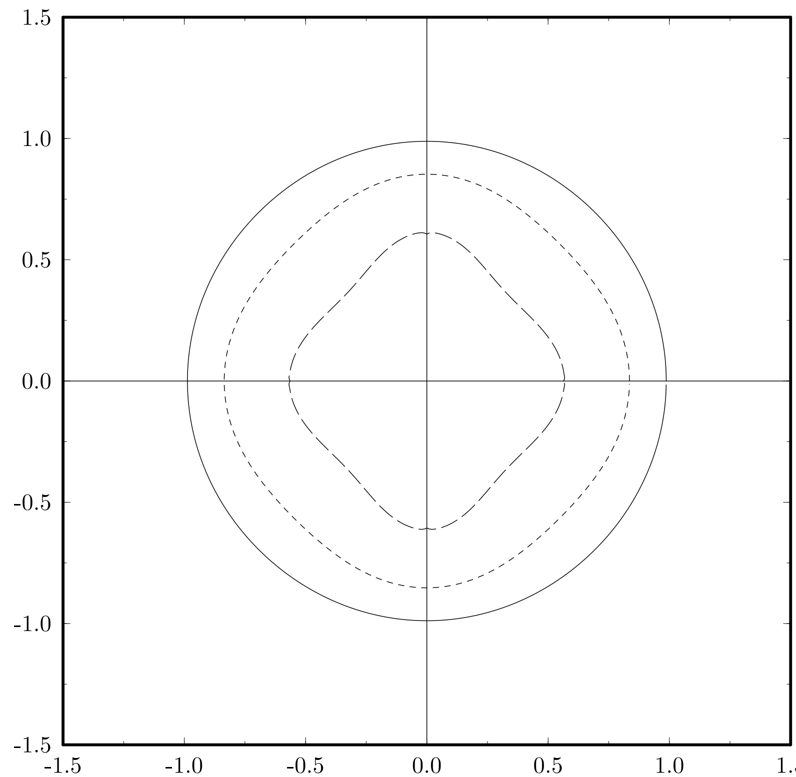


$$p = 2, \quad (-)\ kh = \pi/2, \quad (\cdots\cdots)\ kh = 0.8\pi, \quad (--)\ kh = \pi$$

Two dimensions

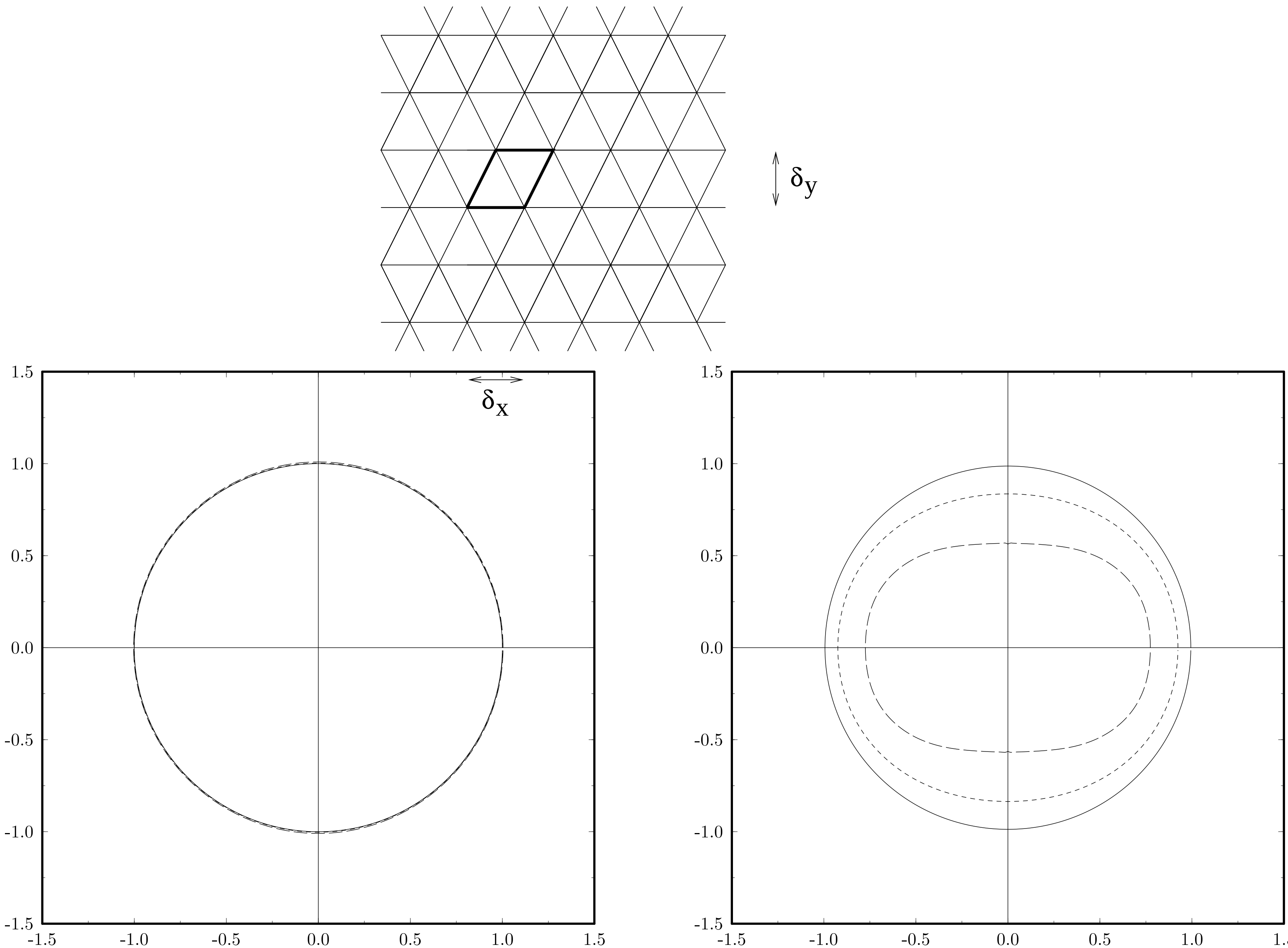






$$p = 2, \quad (-)\ kh = \pi/2, \quad (\cdots\cdots)\ kh = 0.8\pi, \quad (--)\ kh = \pi$$

Two dimensions



$p = 2, \quad (-)\ kh = \pi/2, \quad (\cdots\cdots)\ kh = 0.8\pi, \quad (--)\ kh = \pi$

Boundary Element Method

Helmholtz equation:

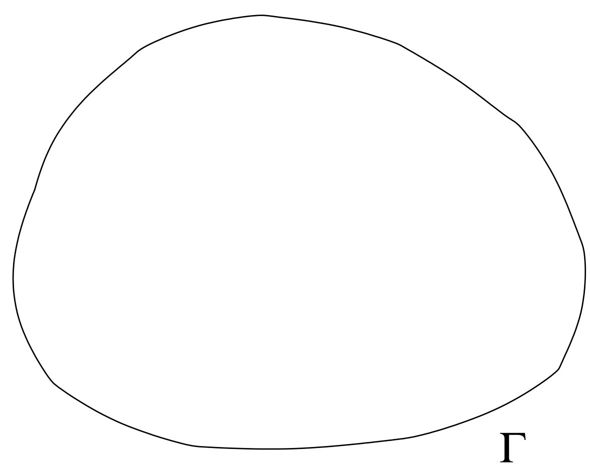
$$\nabla^2\phi + k^2\phi = 0$$

Boundary conditions:

$$\text{Dirichlet BC: } \phi(\mathbf{x}) = b(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma$$

$$\text{Neumann BC: } \frac{\partial\phi}{\partial n}(\mathbf{x}) = b(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma$$

$$\text{Robin BC: } \alpha\phi(\mathbf{x}) + \beta\frac{\partial\phi}{\partial n}(\mathbf{x}) = b(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma$$



Boundary Integral Equation

Direct formulation

$$\phi(\mathbf{x}) = \int_\Gamma \left(G(\mathbf{x},\mathbf{x}_s)\frac{\partial\phi}{\partial n}(\mathbf{x}_s) - \phi(\mathbf{x}_s)\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{x}_s)\right) d\mathbf{x}_s$$

$G =$ fundamental solution

Two dimensions:

$$G(\mathbf{x},\mathbf{x}') = \frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{x}'|)$$

Three dimensions:

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}e^{ik|\mathbf{x}-\mathbf{x}'|}$$

Boundary Integral Equation

$$c_s\phi(\mathbf{x}'_s) = \int_\Gamma \left(G(\mathbf{x}'_{\mathrm{s}},\mathbf{x}_s)\frac{\partial\phi}{\partial n}(\mathbf{x}_s) - \phi(\mathbf{x}_s)\frac{\partial G}{\partial n}(\mathbf{x}'_s,\mathbf{x}_s)\right) d\mathbf{x}_s$$

$\Rightarrow$Advantage: reduce the dimension of the problem by one

Boundary Integral Equation

Combined formulation

$$\phi(\mathbf{x}) = \int_\Gamma \left(\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}_s) - i\eta G(\mathbf{x}, \mathbf{x}_s) \right) f(\mathbf{x}_s) d\mathbf{x}_s$$

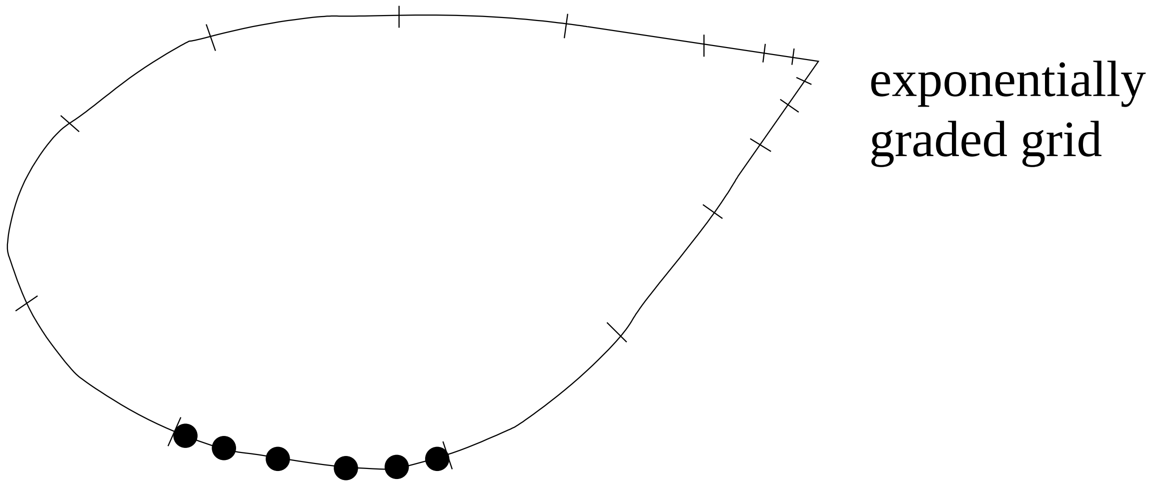
Boundary Integral Equation for Dirichlet BC:

$$c_s f(\mathbf{x}'_s) + \int_\Gamma \left(\frac{\partial G}{\partial n}(\mathbf{x}'_s, \mathbf{x}_s) - i\eta G(\mathbf{x}'_\mathbf{s}, \mathbf{x}_s) \right) f(\mathbf{x}_s) d\mathbf{x}_s = b(\mathbf{x}'_s)$$

Boundary Integral Equation for Neumann BC:

$$c_s \frac{i\eta}{2} f(\mathbf{x}'_s) + \int_\Gamma \left(\frac{\partial^2 G}{\partial n' \partial n}(\mathbf{x}'_s, \mathbf{x}_s) - i\eta G(\mathbf{x}'_s, \mathbf{x}_s) \right) f(\mathbf{x}_s) d\mathbf{x}_s = b(\mathbf{x}'_s)$$

Spectral Collocation Method



$$\text{On element } \Gamma_i,\ \bar{\mathbf{x}}_s = \mathbf{r}_i(t),\ \phi = \sum_{n=0}^{P} u_i^{(n)} \phi_n(t)$$

Basis functions (Chebychev polynomials): $\phi_n(t) = \cos(n \arccos t)$

Collocation points (Gauss-Chebychev): $t_\ell = \cos\left(\frac{2\ell+1}{2P+2}\pi\right), \ell = 0, 1, 2, ..., P$

Mean flow effect

$$(-ik + U\nabla)^2\phi - k^2\phi = 0$$

REF:

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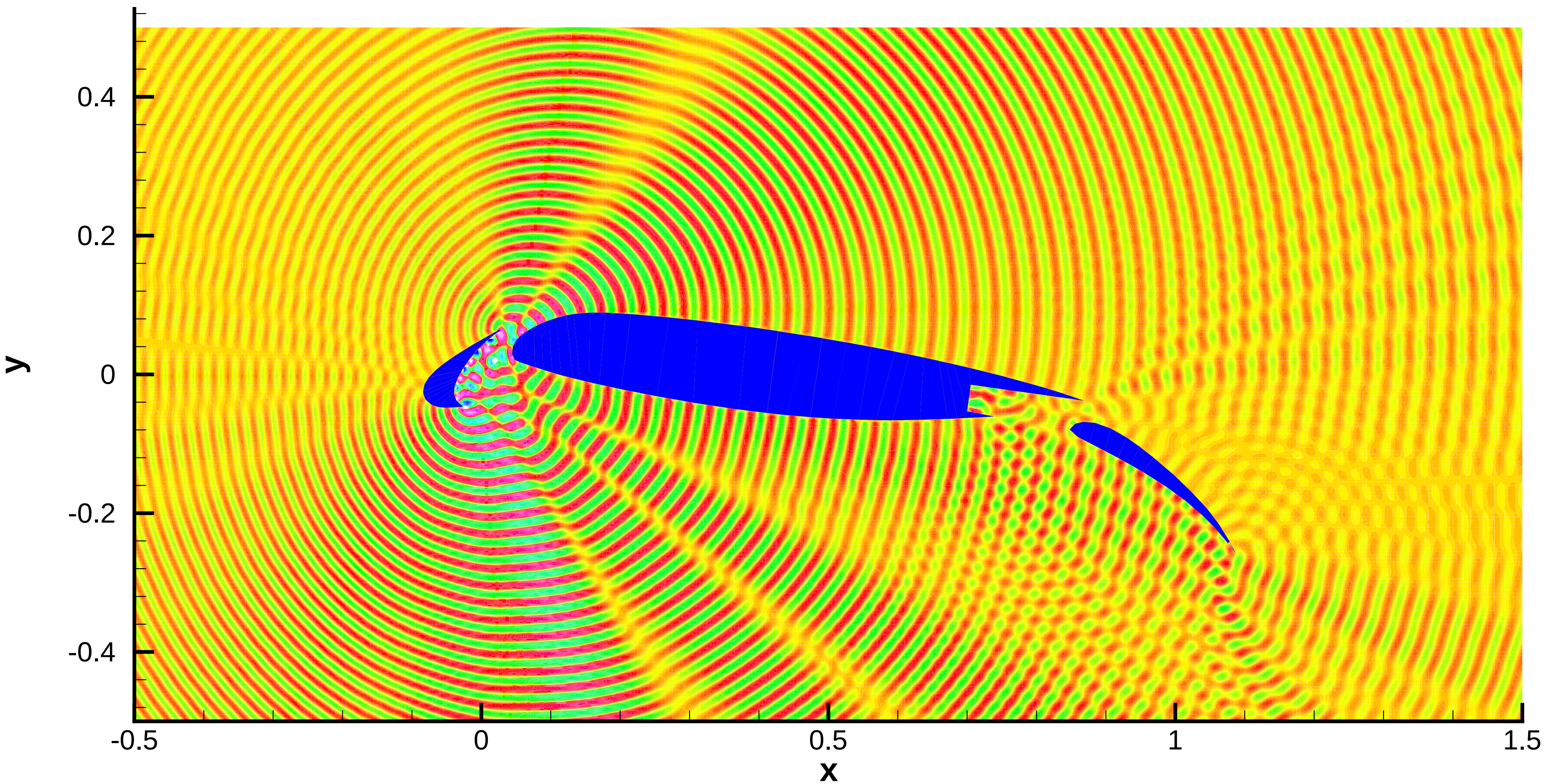
Three element airfoil



$k = 250$, $U = 0.2$, N=226, basis order $p = 5$

Other topics

- ♦ Finite volume and spectral methods
- ♦ Mesh design
- ♦ Boundary Conditions
- ♦ ······

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