Lagrangians Methods for CFD in plasma physics

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and UPMC

1. Context : numerical methods for ICF
2. the Eulerian case as Lagrange+remapp (today)
3. The real 2D Lagrangian case (tomorrow)
4. Conclusion

I present the result of years of investigation at the CEA with many students and collaborators. I focus on Lagrangian schemes for historical reasons. The presentation is split in two parts.

Part I: 1D Lagrange models and eulerian schemas as Lagrange+remapp schemes

Outline

a) A general framework for many models

b) Application to 1D lagrangian gas dynamics, T_i-T_e model ideal MHD (plasma physics with shocks).

c) High order extension: DGM/Reconstruction. Aeroacoustic.

d) Conclusion and perspectives

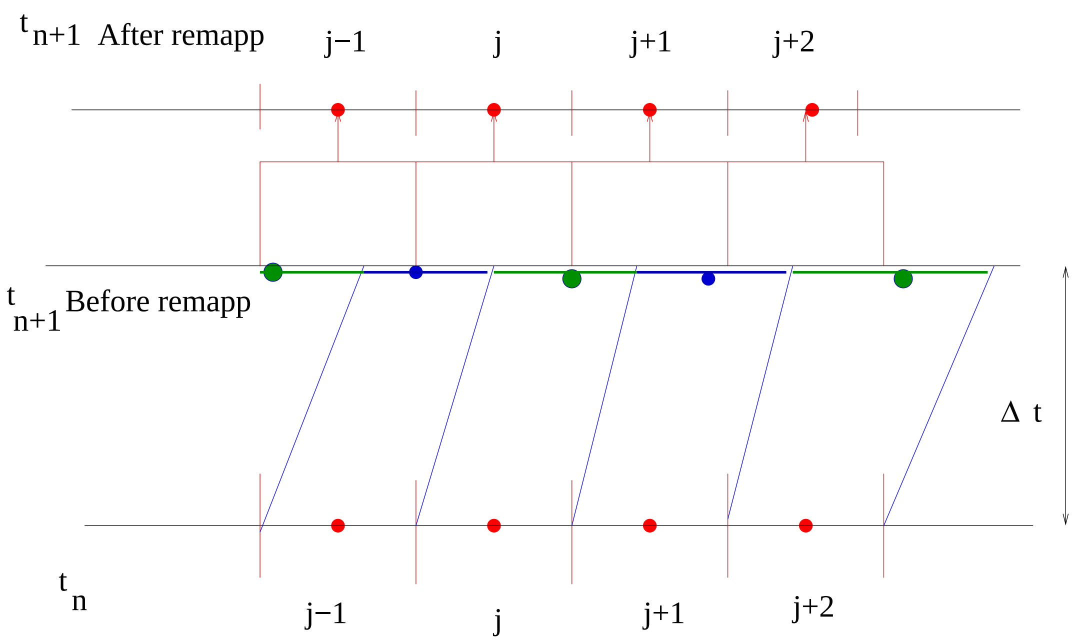


Figure 1: Consider that we use a directional splitting (ADI). In each direction: First we solve in the comobile frame that moves with the fluid : the mesh moves. Then we remapp, that is we project on the old mesh. Remapping is conceptually easy. Even if high order is mandatory to get good results. Let focus on the Lagrange step

1D Euler
$$\begin{cases} \partial_t \rho + \partial_x \rho u & = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) & = 0, \quad p = p(\varepsilon, \rho), \\ \partial_t \rho e + \partial_x (\rho e u + p u) & = 0, \quad e = \frac{1}{2} u^2 + \varepsilon, \\ \partial_t \rho S + \partial_x (\rho S u) & \geq 0. \end{cases}$$

Let define $x = x(m,t)$: $\frac{dx}{dt} = u(x,t), \quad x(X,0) = X$. The mass variable is $\rho(X,0)dX = dm, \quad X = X(m)$.

The Lagrangian equations are

$$\begin{cases} \partial_t \overline{\tau} - \partial_m \overline{u} = 0, \\ \partial_t \overline{u} + \partial_m \overline{p} = 0, \\ \partial_t \overline{e} + \partial_m \overline{pu} = 0, \\ \partial_t \overline{S} \geq 0, \end{cases} \qquad \begin{cases} \overline{\tau}(m,t) = \rho^{-1}(x(m,t),t), \\ \overline{u}(m,t) = u(x(m,t),t), \\ \overline{e}(m,t) = e(x(m,t),t), \\ \overline{S}(m,t) = S(x(m,t),t). \end{cases}$$

This is the good system to work on for the Lagrangian step.

Let $\partial_t U + \partial_m f(U) = 0$, $U \in R^n$ and $f(U) \in R^n$. We assume that there exists a strictly concave physical entropy $S \in R$ with a vanishing entropy flux: $\partial_t S = 0 \quad \forall$ smooth U.

Main Theoretical result: All systems of conservation laws with vanishing entropy flux coming from the mechanics ("fluid models", Galilean invariance, reversibility for smooth solutions) have the form

$$\partial_t U + \partial_m \begin{pmatrix} M\Psi \\ -\frac{1}{2}\Psi^t M \Psi \end{pmatrix} = 0$$

where $M = M^t$ is a $n-1 \times n-1$ constant symmetric matrix, $V = \nabla_U S =$, $\Psi = \left(\frac{V_1}{V_n} \frac{V_2}{V_n} \ldots \frac{V_{n-1}}{V_n}\right)$, $U_n = e$ is the total energy and $V_n = \frac{1}{T}$.

$\Psi^t M \Psi = (\Psi, M\Psi) \in \mathbb{R}$.

Example

Euler equations in Lagrange coordinates

$$\partial_t \left| \begin{array}{c} \tau \\ u \\ e \end{array} \right. + \partial_m \left| \begin{array}{c} -u \\ p \\ pu \end{array} \right. = 0$$

corresponds to $dS = \frac{1}{T}(d\varepsilon + pd\tau) = \frac{1}{T}(de - udu + pd\tau)$. So

$$\psi = \begin{pmatrix} p \\ -u \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In the general case $M = \begin{pmatrix} 0 & M_1 \\ M_1^t & 0 \end{pmatrix}$ where M_1 is rectangular.

For such a system one has $V = \nabla_U S = V_n(\Psi, 1)$. So

$$\partial_t S = V_n \begin{pmatrix} \Psi \\ 1 \end{pmatrix} .\partial_x \begin{pmatrix} M\Psi \\ -\frac{1}{2}\Psi^t M\psi \end{pmatrix}$$

$$= V_n \left((\Psi, M\partial_m \Psi) - (\Psi, M\partial_m \Psi) \right) = 0.$$

The entropy is constant along integral lines. This is true for many systems (MHD, ionized gas, Lorentzian gas dynamics, ...).

This is the type of systems I want to study

Consequence for numerical schemes

Consider the explicit scheme
$\frac{\Delta m}{\Delta t}(U_i^{n+1} - U_i^n) + f(U)_{i+\frac{1}{2}} - f(U)_{i-\frac{1}{2}} = 0$ with

$$f(U) = \begin{pmatrix} M\Psi \\ -\frac{1}{2}\Psi^t M\Psi \end{pmatrix}.$$

We introduce the splitting of M in a symmetric positive part and a symmetric negative part: $M = M^+ + M^-$, $M^+ = (M^+)^t \geq 0$ and $M^- = (M^-)^t \leq 0$.

Many non expensive splittings are available. The explicit flux is

$$f(U)_{i+\frac{1}{2}} = \begin{pmatrix} M^+_{i+\frac{1}{2}}\Psi_{i+1} + M^-_{i+\frac{1}{2}}\Psi_i \\ -\frac{1}{2}(\Psi^n_{i+1}, M^+_{i+\frac{1}{2}}\Psi_{i+1}) - \frac{1}{2}(\Psi_i, M^-_{i+\frac{1}{2}}\Psi_i) \end{pmatrix}$$

The flux is consistent. If $U_i^n = U_{i+1}^n$ then $f(U)_{i+\frac{1}{2}}$ is equal to

$$f(U)_{i+\frac{1}{2}} = f(U_i^n) = f(U_{i+1}^n).$$

Numerical stability from basic physical principles The scheme is entropy consistent under CFL condition, i.e. there exists constants $c_i^n > 0$ such that

$$\text{if } c_i^n \frac{\Delta t}{\Delta x} \leq 1, \text{ then } S(U_i^{n+1}) \geq S(U_i^n)$$

For gas dynamics $S = \log(\varepsilon \tau^{\gamma-1})$. The inequality is a non linear stability result for the Lagrange step of the scheme. Since the remapp is stable, the Lagrange+remapp scheme is stable and very robust for a large variety of models.

A simple proof. Consider the semi-discrete scheme $\Delta m \frac{d}{dt} U_i + f(U)_{i+\frac{1}{2}} - f(U)_{i-\frac{1}{2}} = 0$. Compute ($V_n = \frac{1}{T}$)

$$T \Delta m \frac{d}{dt} S_i = -\left((\Psi_i, 1), f(U)_{i+\frac{1}{2}} - f(U)_{i-\frac{1}{2}} \right)$$

$$= -\left((\Psi_i, M^+_{i+\frac{1}{2}} \Psi_{i+1}) + (\Psi_i, M^-_{i+\frac{1}{2}} \Psi_i) \right.$$

$$\left. -\frac{1}{2}(\Psi_{i+1}, M^+_{i+\frac{1}{2}} \Psi_{i+1}) - \frac{1}{2}(\Psi_i, M^-_{i+\frac{1}{2}} \Psi_i) \right) + (...)$$

$$= -\left((\Psi_i, M^+_{i+\frac{1}{2}} \Psi_{i+1}) - \frac{1}{2}(\Psi_{i+1}, M^+_{i+\frac{1}{2}} \Psi_{i+1}) - \frac{1}{2}(\Psi_i, M^+_{i+\frac{1}{2}} \Psi_i) \right.$$

$$\left. +(\Psi_i, M^-_{i+\frac{1}{2}} \Psi_i) - \frac{1}{2}(\Psi_i, M^-_{i+\frac{1}{2}} \Psi_i) - \frac{1}{2}(\Psi_i, M^-_{i+\frac{1}{2}} \Psi_i) \right) + \left(... - \frac{1}{2}(\Psi_i, M \Psi_i) \right)$$

$$= (\geq 0) - (= 0) - (= 0) - (\leq 0) \geq 0.$$

Part I b)

1 Physical examples: gas dynamics, T_i-T_e plasma, MHD, 2D AMR MHD.

2 High order extension: DGM versus reconstruction plus limiting

3 Very high on cartesian grids finite volume scheme for the acoustic approximation: collaboration with Pascal Havé, Stéphane Del Pino and Hervé Jourdren.

The simplest example : 1D gas dynamics

Choose a coefficient $\alpha_{i+\frac{1}{2}} = (\rho^* c^*)_{i+\frac{1}{2}} > 0$. We split M into

$$M^+_{i+\frac{1}{2}} = \begin{pmatrix} \frac{1}{2\alpha_{i+\frac{1}{2}}} & \frac{1}{2} \\ \frac{1}{2} & \frac{\alpha_{i+\frac{1}{2}}}{2} \end{pmatrix} \text{ and } M^-_{i+\frac{1}{2}} = \begin{pmatrix} -\frac{1}{2\alpha_{i+\frac{1}{2}}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\alpha_{i+\frac{1}{2}}}{2} \end{pmatrix}.$$

The difference equations are

$$\frac{\Delta m}{\Delta t} \begin{pmatrix} \tau_i^{n+1} - \tau_i^n \\ u_i^{n+1} - u_i^n \\ e_i^{n+1} - e_i^n \end{pmatrix} + \begin{pmatrix} -u^*_{i+\frac{1}{2}} + u^*_{i-\frac{1}{2}} \\ p^*_{i+\frac{1}{2}} - p^*_{i-\frac{1}{2}} \\ (pu)^*_{i+\frac{1}{2}} - (pu)^*_{i-\frac{1}{2}} \end{pmatrix} = 0$$

where

$$\begin{cases} p^*_{i+\frac{1}{2}} = \frac{1}{2}(p_i^n + p_{i+1}^n) + \frac{(\rho^* c^*)_{i+\frac{1}{2}}}{2}(u_i^n - u_{i+1}^n) \\ u^*_{i+\frac{1}{2}} = \frac{1}{2}(u_i^n + u_{i+1}^n) + \frac{1}{2(\rho^* c^*)_{i+\frac{1}{2}}}(p_i^n - p_{i+1}^n) \end{cases}.$$

This is the acoustic approximation. It may be recovered through a standard analysis of the linearized Riemann invariants Richtmyer-Morton (1957), Godounov (1959), ... HLLE, ... These linearized Riemann invariants are $dp \pm (\rho c) du = 0$ where (ρc) is the product of the density times the velocity of sound $(\rho c)^2 = -\frac{\partial p}{\partial \tau | S}$. We freeze $(\rho c) = (\rho^* c^*)_{i+\frac{1}{2}}$.

The solution of the linearized Riemann problem between a left state τ_i^n, u_i^n, e_i^n and a right state $\tau_{i+1}^n, u_{i+1}^n, e_{i+1}^n$ gives an intermediate state referred as p^*, u^*. The equation for these p^*, u^* is

$$\begin{cases} p^* + (\rho^* c^*)_{i+\frac{1}{2}} u^* = p_i^n + (\rho^* c^*)_{i+\frac{1}{2}} u_i^n \\ p^* - (\rho^* c^*)_{i+\frac{1}{2}} u^* = p_{i+1}^n - (\rho^* c^*)_{i+\frac{1}{2}} u_{i+1}^n \end{cases}$$

whose solution is exactly the acoustic flux.

However many other splittings are available, some of them being exotic.

One may choose to split M into ($\alpha_{i+\frac{1}{2}} = (\rho^* c^*)_{i+\frac{1}{2}} > 0$)

$$M^+_{i+\frac{1}{2}} = \begin{pmatrix} \frac{1}{\alpha_{i+\frac{1}{2}}} & 1 \\ 1 & \alpha_{i+\frac{1}{2}} \end{pmatrix} \text{ and } M^-_{i+\frac{1}{2}} = \begin{pmatrix} -\frac{1}{\alpha_{i+\frac{1}{2}}} & 0 \\ 0 & -\alpha_{i+\frac{1}{2}} \end{pmatrix}.$$

We end up to other formulas for the fluxes

$$\begin{cases} p^*_{i+\frac{1}{2}} = p^n_{i+1} + (\rho^* c^*)_{i+\frac{1}{2}}(u^n_i - u^n_{i+1}) \\ u^*_{i+\frac{1}{2}} = u^n_{i+1} + \frac{1}{(\rho^* c^*)_{i+\frac{1}{2}}}(p^n_i - p^n_{i+1}) \end{cases}.$$

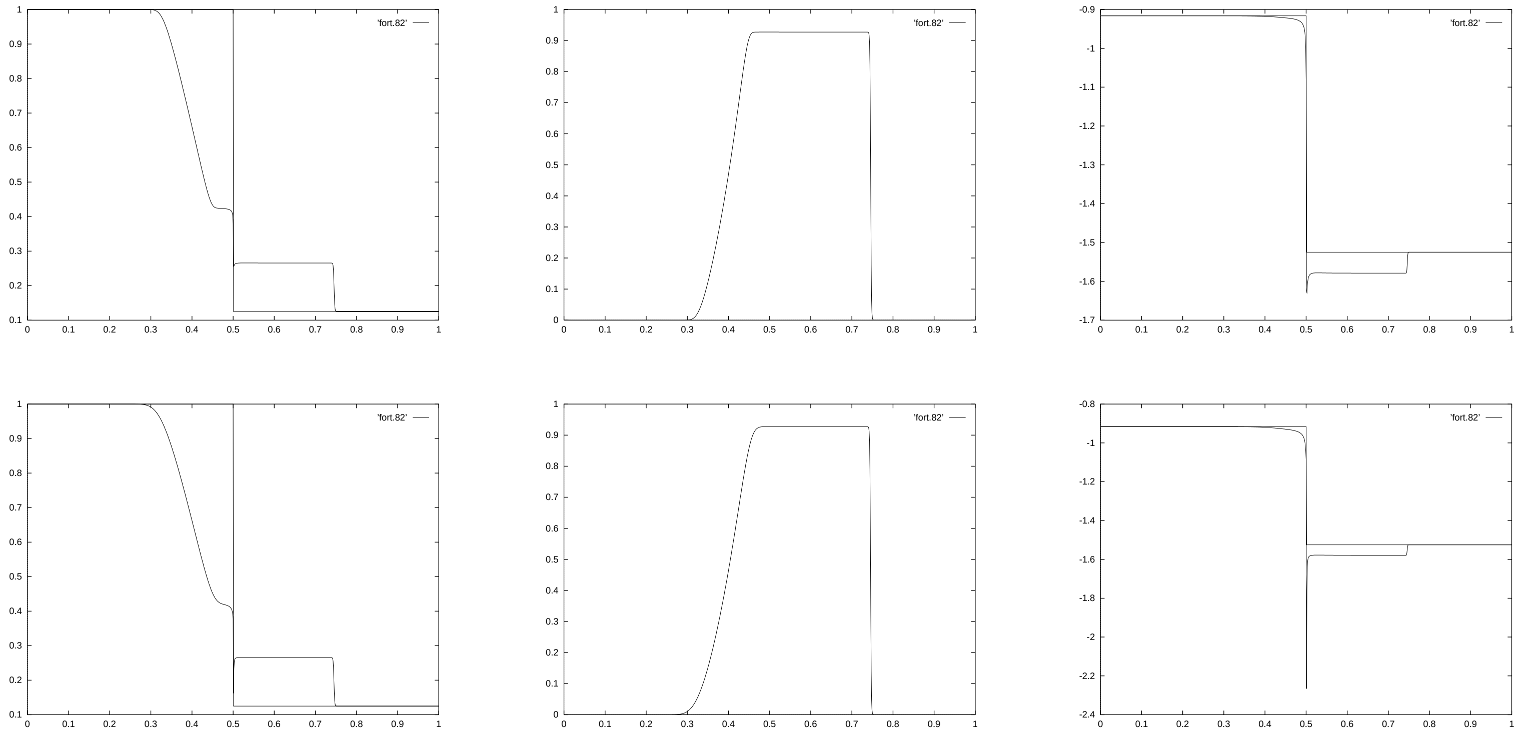


Figure 2: ρ, u, $-S$ for the Sod shock tube at time $t = 0.14$ in pure Lagrange. Acoustic (top) versus exotic (bottom). Entropic schemes are stable. They converge to the correct solution. The difference is more a matter of accuracy.

Application to $T_i - T_e$ model

In view of solving ICF oriented problems we introduce the non-conservative $T_i - T_e$ model for ionized gas, in Euler coordinates

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p_i + p_e) = 0 \\ \partial_t \rho \varepsilon_i + \partial_x(\rho u \varepsilon_i) + p_i \partial_x u = \frac{1}{\tau_{ei}}(T_e - T_i) \\ \partial_t \rho \varepsilon_e + \partial_x(\rho u \varepsilon_e) + p_e \partial_x u = \frac{1}{\tau_{ei}}(T_i - T_e) + \partial_x K_e \partial_x T_e \end{cases}.$$

The density of total energy $e = \varepsilon_i + \varepsilon_e + \frac{1}{2}u^2$ satisfies

$$\partial_t \rho e + \partial_x(\rho u e + p_i u + p_e u) = \partial_x K_e \partial_x T_e.$$

The correct equation for electrons is

$$\partial_t \rho S_e + \partial_x \rho u S_e = \frac{1}{T_e \tau_{ei}}(T_i - T_e) + \frac{1}{T_e} \partial_x K_e \partial_x T_e.$$

So the system we want to solve is

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p_i + p_e) = 0 \\ \partial_t \rho e + \partial_x(\rho u e + p_i u + p_e u) = \partial_x K_e \partial_x T_e \\ \partial_t \rho S_e + \partial_x \rho u S_e = \frac{1}{T_e \tau_{ei}}(T_i - T_e) + \frac{1}{T_e}\partial_x K_e \partial_x T_e \end{cases} .$$

First we solve the hydrodynamic part

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p_i + p_e) = 0 \\ \partial_t \rho e + \partial_x(\rho u e + p_i u + p_e u) = 0 \\ \partial_t \rho S_e + \partial_x \rho u S_e = 0 \end{cases} \iff \begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m(p_i + p_e) = 0 \\ \partial_t S_e = 0 \\ \partial_t e + \partial_m(p_i u + p_e u) = 0 \end{cases} .$$

The structure of the lagrangian flux is given by

$$\psi = \begin{pmatrix} p = p_i + p_e \\ -T_e \\ -u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We get an entropy consistent scheme such that $(S_{\text{ion}})_j^L \geq (S_{\text{ion}})_j^n$.

Finally after remapping+ solving the right hand side we get a stable, conservative and globally entropy consistent scheme for the $T_i - T_e$ model and for a very large set of equations of state p_e, p_i and coefficients τ_{ei}, K_e.

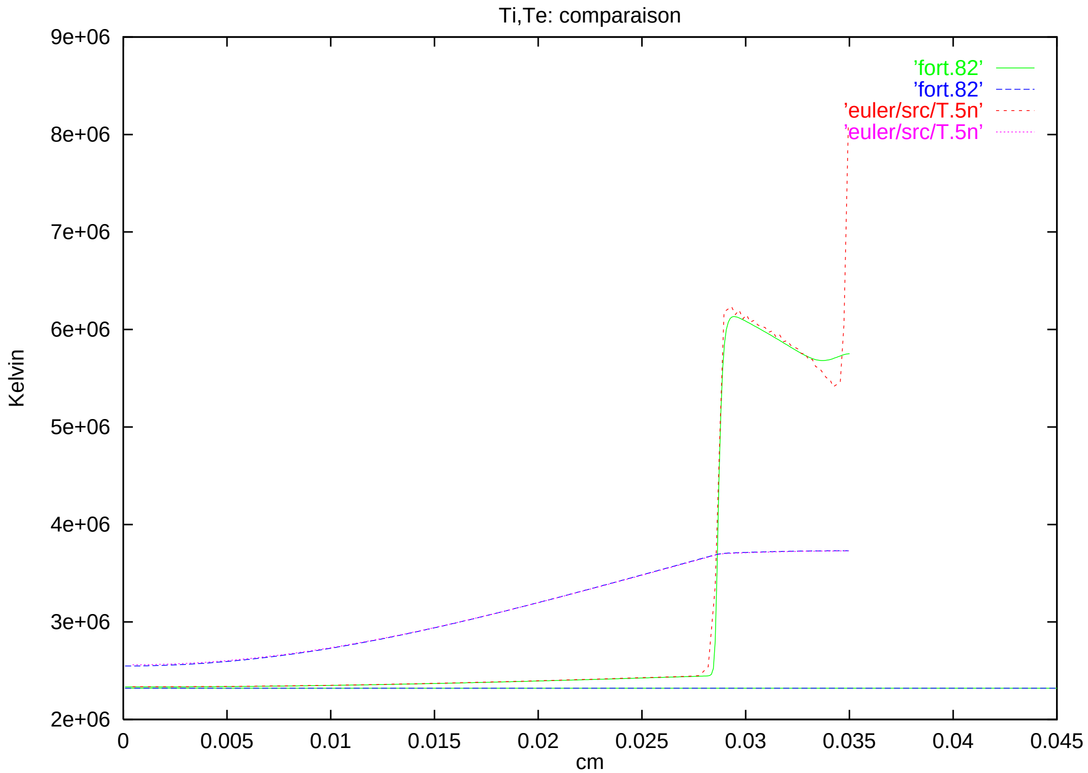


Figure 3: $T_i - T_e$ model. Comparison with the VNR scheme. The piston on the right models the laser that pushes the target. The shock is corrrect because the entropy of the scheme is on the ions

Be carefull that

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t\rho u+\partial_x(\rho u^2+p_i+p_e)=0\\ \partial_t\rho e+\partial_x(\rho ue+p_iu+p_eu)=0\\ \partial_t\rho S_e+\partial_x\rho uS_e=0\end{array}\right. \Longleftrightarrow \left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t\rho u+\partial_x(\rho u^2+p_i+p_e)=0\\ \partial_t\rho e+\partial_x(\rho ue+p_iu+p_eu)=0\\ \partial_t\rho S_i+\partial_x\rho uS_i=0\end{array}\right.$$

for smoth solutions. But for non smooth solutions (shocks) this is no more equivalent.

The ideal multi-D MHD (strong coupling between a plasma and the electro-magnetic field) system written in conservative form is

$$\partial_t \begin{pmatrix} \rho \\ B \\ \rho u \\ E \end{pmatrix} + \begin{pmatrix} \nabla.(\rho u) \\ \nabla.(u \otimes B - B \otimes u) \\ \nabla.(\rho u \otimes u - \frac{B \otimes B}{\mu}) + \nabla P \\ \nabla.((E+P)u - \frac{B}{\mu}(u.B)) \end{pmatrix} = 0.$$

New difficulty : the conservative formulation of MHD is not hyperbolic (in the general mathematical sense). However it is **linearly well posed** is one do not forget about the fre divergence constraint on the magnetic field. Simple calculations show that $\nabla.B = 0$ is preserved by the system. The equation for entropy is

$$\partial_t(\rho S) + \nabla.(\rho u S) \geq -\frac{(B.u)}{\mu \rho T}\nabla.B = 0.$$

1D lagrange for MHD is hyperbolic and conservative. We obtain

$$\partial_t \begin{pmatrix} \tau \\ \tau B_y \\ \tau B_z \\ u \\ v \\ w \\ e \end{pmatrix} + \partial_m \begin{pmatrix} -u \\ -B_x v \\ -B_x w \\ P^* \\ -\frac{B_x}{\mu} B_y \\ -\frac{B_x}{\mu} B_z \\ P^* u - \frac{B_x}{\mu}(v B_y + w B_z) \end{pmatrix} = 0$$

with

$$P^* = p + \frac{1}{2\mu}(-B_x^2 + B_y^2 + B_z^2)$$

The physical entropy $S(\varepsilon, \tau)$ gives the mathematical entropy $-S$.

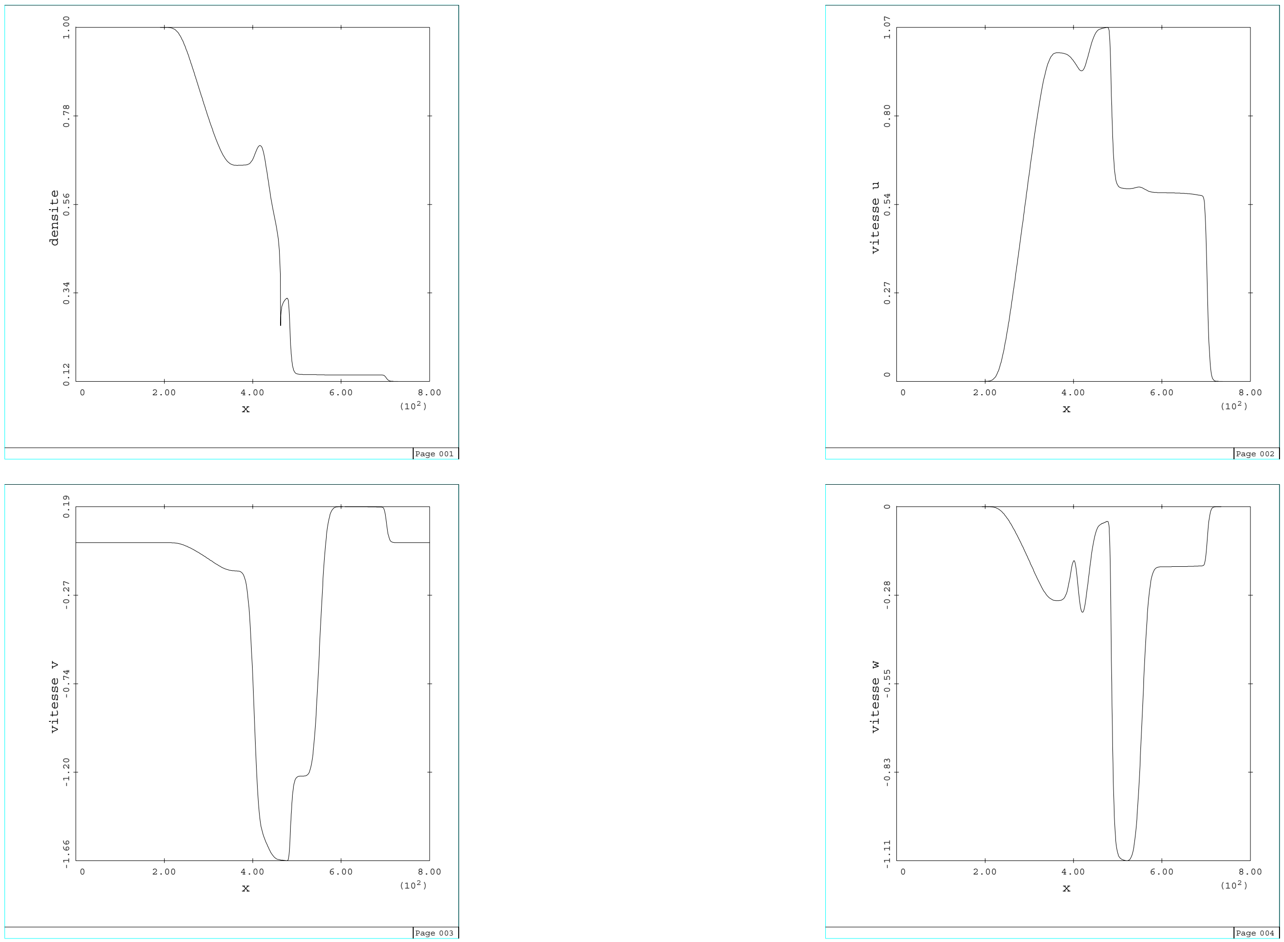


Figure 4: Shock tube. If $B_x^2 + B_y^2 + B_z^2 \neq 0$ then the scheme (with F. Bézard, JCP, 1998) is not equal to the Roe scheme.

Part I b)

The Landau-Godunov-Roe-Powell formulation is hyperbolic

$$\partial_t \begin{pmatrix} \rho \\ B \\ \rho u \\ E \end{pmatrix} + \begin{pmatrix} \nabla.(\rho u) \\ \nabla.(u \otimes B - B \otimes u) \\ \nabla.(\rho u \otimes u - \frac{B \otimes B}{\mu}) + \nabla P \\ \nabla.((E+P)u - \frac{B}{\mu}(u.B)) \end{pmatrix} = - \begin{pmatrix} 0 \\ u \\ \frac{B}{\mu} \\ \frac{(u.B)}{\mu} \end{pmatrix} \nabla.B.$$

But is not conservative. Our goal was to derive a stable, conservative, entropic scheme.

We use another formulation

$$\partial_t \begin{pmatrix} \rho \\ B \\ \rho u \\ E \end{pmatrix} + \begin{pmatrix} \nabla.(\rho u) \\ \nabla.(\rho u \otimes B - C \otimes u) \\ \nabla.(\rho u \otimes u - \frac{C \otimes B}{\mu}) + \nabla P \\ \nabla.((E+P)u - \frac{C}{\mu}(u.B)) \end{pmatrix} = 0.$$

where C is an exterior field such that $\nabla.C = 0$. In practice one solves

$$\partial_t C + \nabla \wedge (u \wedge C) = 0$$

with an ad-hoc method that guarantees $\nabla.C = 0$.

Let $\Psi^t = \left(\ P, -\frac{B}{\mu}, -u \ \right)$. The new abstract multiD quasi-lagrangian formulation is ($\mathrm{D}_t = \partial_t + u.\nabla$)

$$\rho \mathrm{D}_t U + \partial_x f(U) + \partial_y g(U) + \partial_z h(U) = 0 \text{ where } f(U) = \begin{pmatrix} M_1 \Psi \\ -\frac{1}{2}(\Psi, M_1 \Psi) \end{pmatrix}, \cdots,$$

$$\text{and } M_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_i \\ 1 & C_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_i & 0 & 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

On has $M_i = M_i^t$ and

$$\partial_x M_1 + \partial_y M_2 + \partial_z M_3 = 0.$$

We use the lagrange+remapp scheme for $\bullet$ and a direct integration of C.

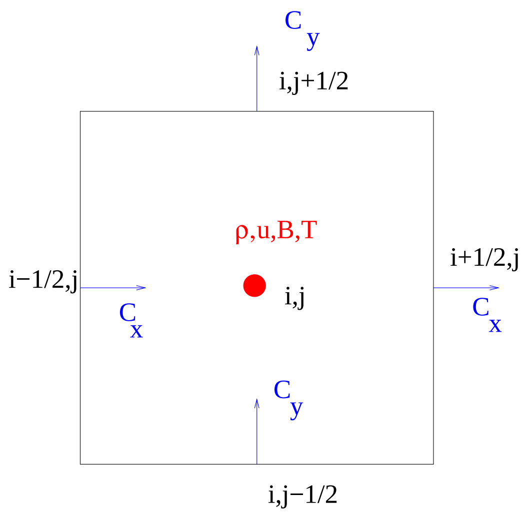
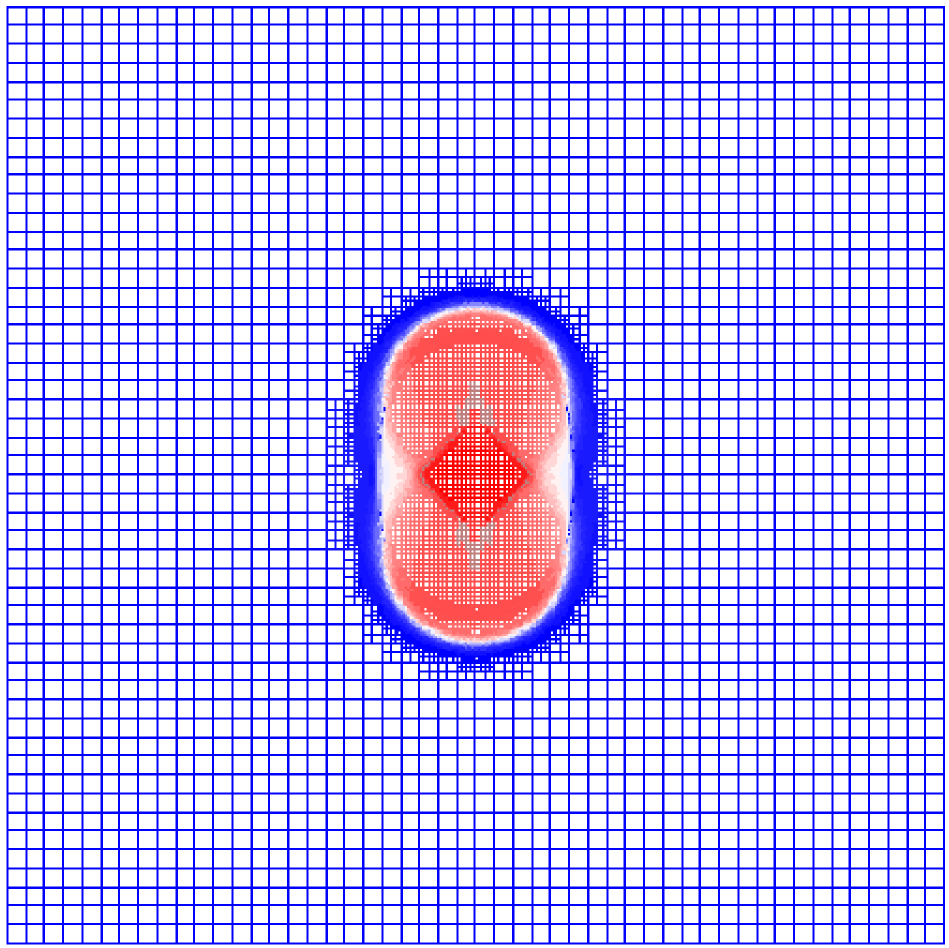


Figure 5: Degrees of freedom. C is discretized with $\partial_t C_x + \partial_y q = 0$ and $\partial_t C_y - \partial_x q = 0$ where q is given at the nodes.

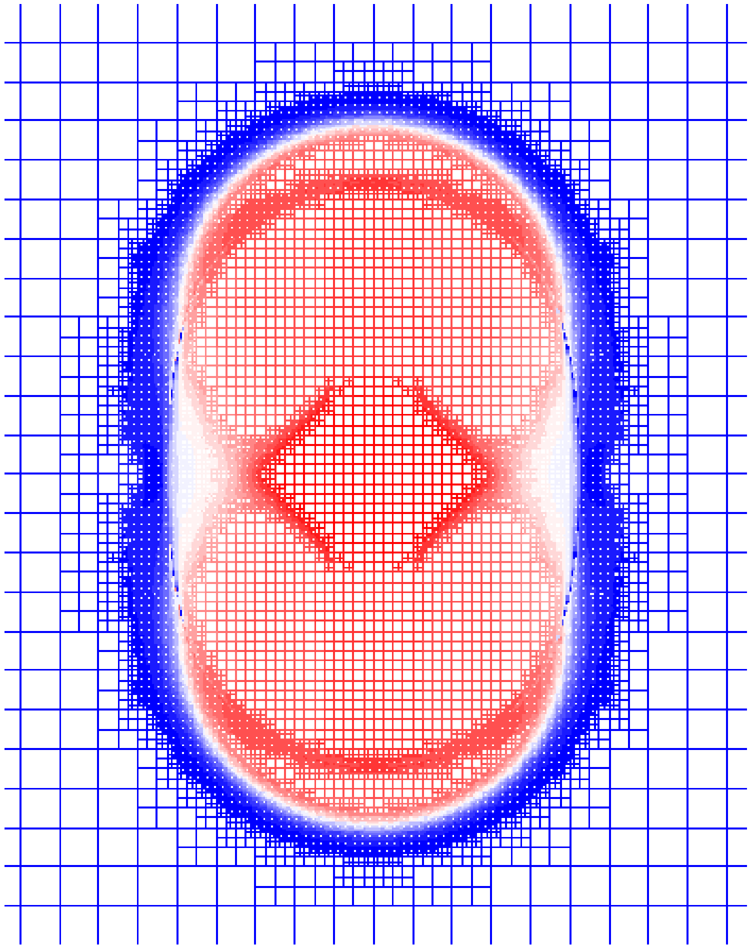
Main result (with F. Desveaux, tech. rep. CMLA) : Assume $C^n_{i+\frac{1}{2},j} - C^n_{i-\frac{1}{2},j} + C^n_{i,j+\frac{1}{2}} - C^n_{i,j+\frac{1}{2}} = 0.$

Then $S^L_{ij} \geq S^n_{ij}$ under CFL.

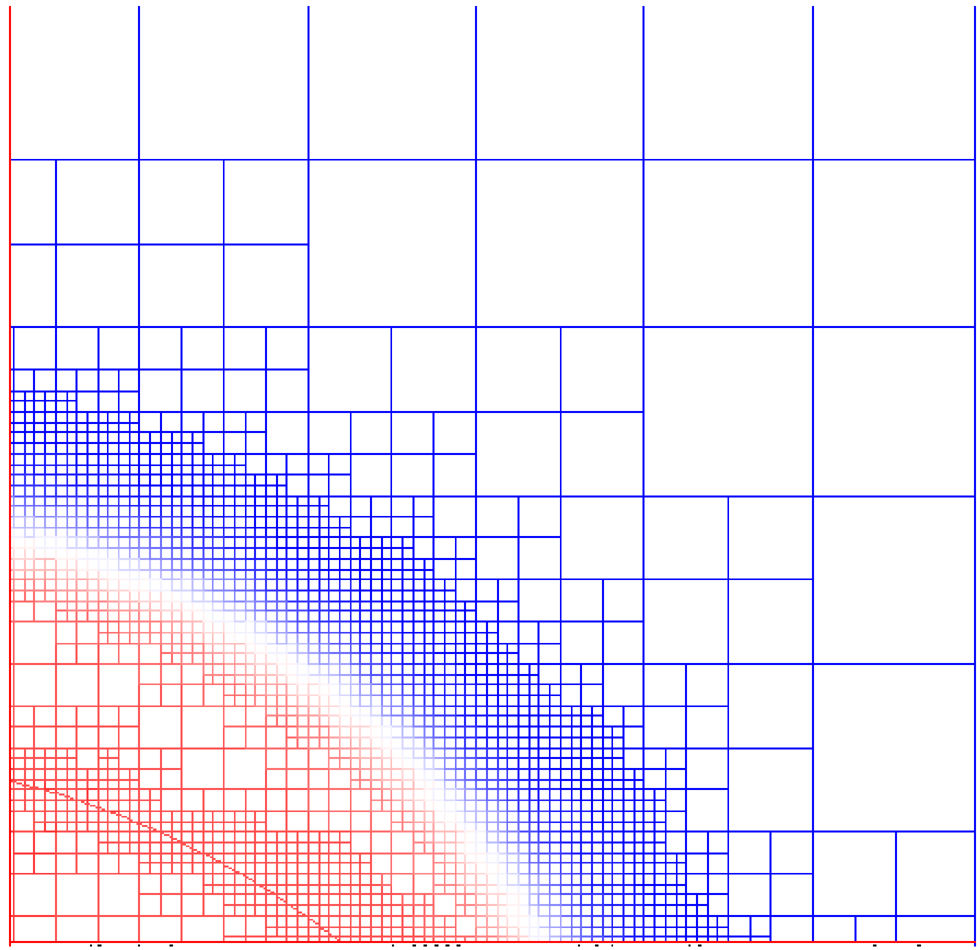
Pressure for the Collela-Malagoni test case



Pressure for the Collela-Malagoni test case : zoom



Pressure for the Collela-Malagoni test case : zoom-zoom



How about high order extension ? It is possible to preserve the discrete entropy inequality in the Lagrange step ?

Two approaches : DGM and reconstruction *à la* Van Leer.

It is possible to preserve the entropy inequality for DGM (B.D. VII conference on hyperbolic problems, Zurich, 1998). But the CPU cost of the method makes it non competitive with respect to reconstruction.

Example of DGM for Euler equations in the axisymmetric case

$$\begin{cases} \partial_t(y^d\rho) + \partial_x(y^d\rho u) + \partial_y(y^d\rho v) = 0, \\ \partial_t(y^d\rho u) + \partial_x(y^d\rho u^2 + y^d p) + \partial_y(y^d\rho uv) = 0, \\ \partial_t(y^d\rho v) + \partial_x(y^d\rho uv) + \partial_y(y^d\rho v^2 + y^d p) = dy^{d-1}p, \\ \partial_t(y^d\rho e) + \partial_x(y^d\rho ue + y^d pu) + \partial_y(y^d\rho ve + y^d pv) = 0. \end{cases}$$

d=0: 2D classical.

d=1: the axisymmetric 2.5D case. Here the equation on v is non conservative. For simplicity, the domain is a square $(x,y) \in \Omega =]0,1[\times]0,1[$, and the pressure provided by a γ-law $p = (\gamma - 1)(\rho e - \rho\frac{u^2+v^2}{2})$.

The Kidder problem It is very strong isentropic convergent flow. The analytical solution is self-similar for $\gamma = \frac{5}{3}$. The solution at (r,t) is related to the solution at $(R, t=0)$ through the transformation $r = R\sqrt{1-\frac{t^2}{\tau^2}}$, this transformation is defined for t smaller than the focusing time $0 \leq t < \tau$. The initial conditions are

$$\begin{cases} \rho(r,0) = \left(\rho_2^{\gamma-1}\frac{r^2-R_1^2}{R_2^2-R_1^2} + \rho_1^{\gamma-1}\frac{R_2^2-r^2}{R_2^2-R_1^2}\right), \\ u(r,0) = 0, \\ \varepsilon(r,0) = \rho(r,0)^{\gamma-1}, \\ p(r,0) = (\gamma-1)\rho(r,0)^{\gamma}, \\ \tau = \sqrt{\frac{1}{2\gamma}\left(\frac{R_2^2-R_1^2}{\rho_2^{\gamma-1}-\rho_1^{\gamma-1}}\right)}, \end{cases}$$

and then

$$\begin{cases} \rho(r,t) = \frac{\rho(R)}{h(t)^3}, \\ u(r,t) = \frac{dr}{dt}, \\ \varepsilon(r,t) = \rho(r,t)^{\gamma-1}, \\ p(r,t) = \frac{\rho(R)}{h(t)^{3\gamma}}. \end{cases}$$

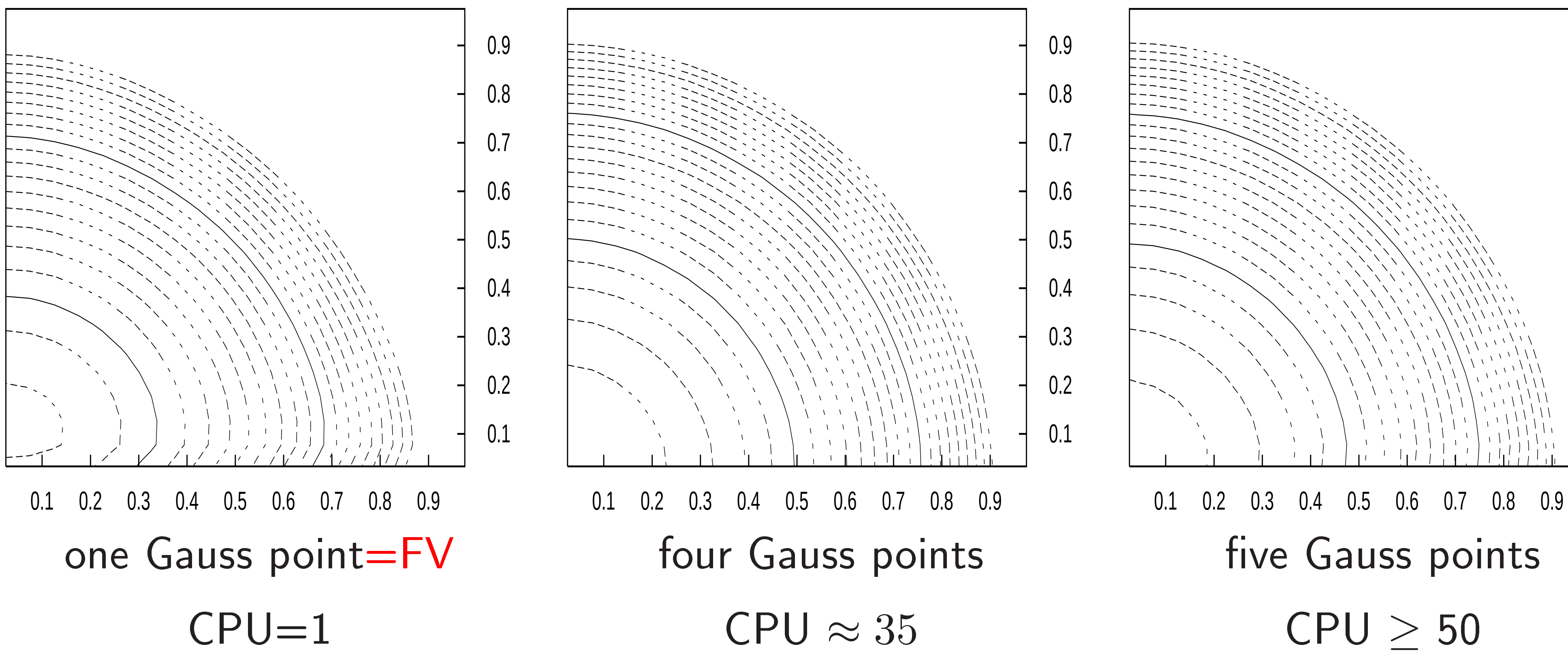


Figure 6: Isolines of the density for the Kidder problem. The best result if with 4 points (Q_1 discontinuous). But the CPU cost is bad

An example of high order reconstruction à la Van Leer (ADER like), collaboration with S. Delpino, H. Jourdren and Pascal Havé (Post-doc at the CEA)

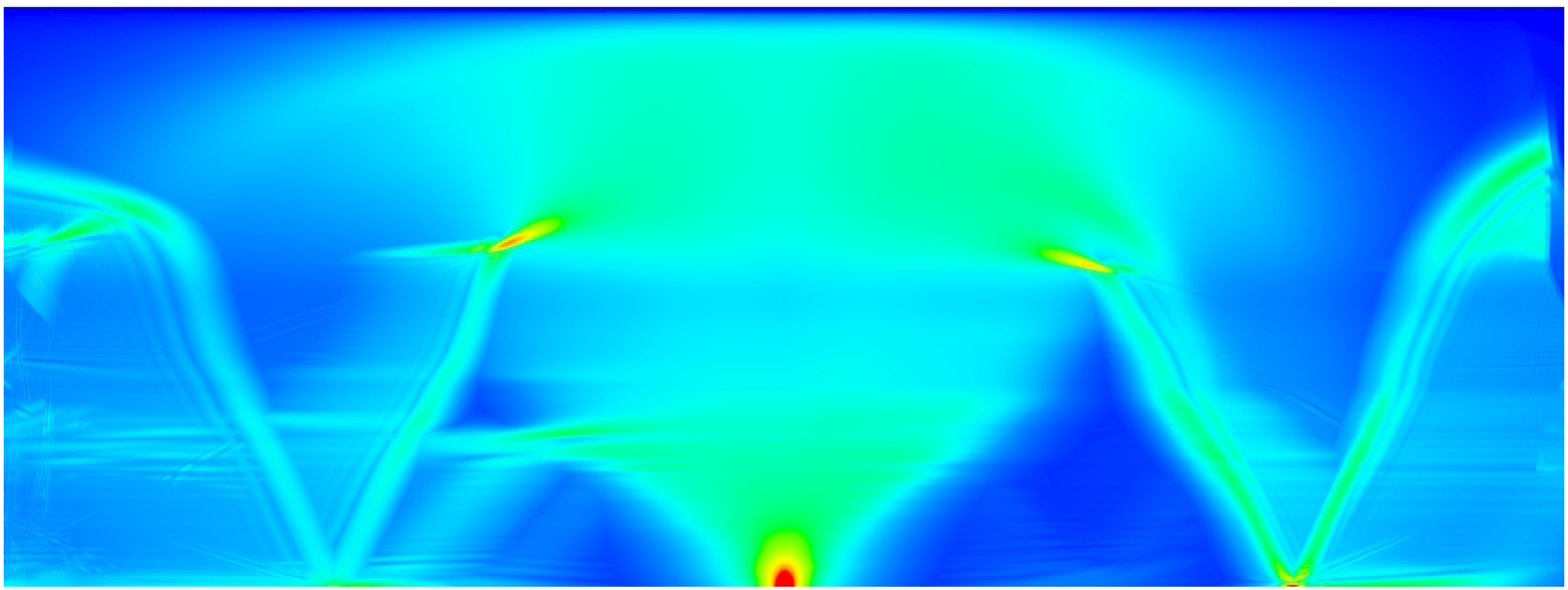


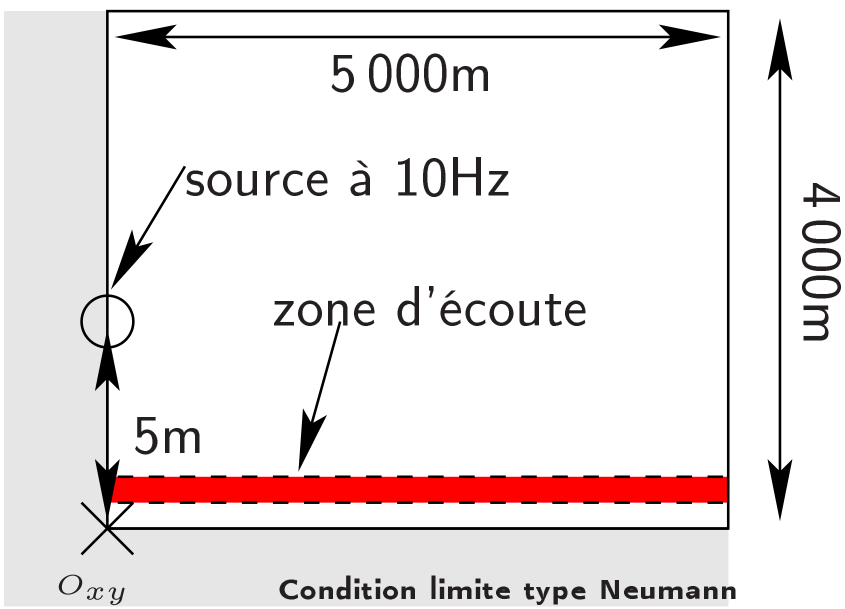
Figure 7: A complete different problem: aeroacoustic in the atmosphere. Maximal pressure in atmosphere

Part I c)

Principle: The acoustic system is the linearized Lagrange system.

Propagation in the air on distances greater than 100 wavelengths

$$\partial_t p + \rho c^2(\partial_x u + \partial_y v) = 0,\ \rho\partial_t u + \partial_x p = 0,\ \rho\partial_t + \partial_y p = 0.$$



Données :

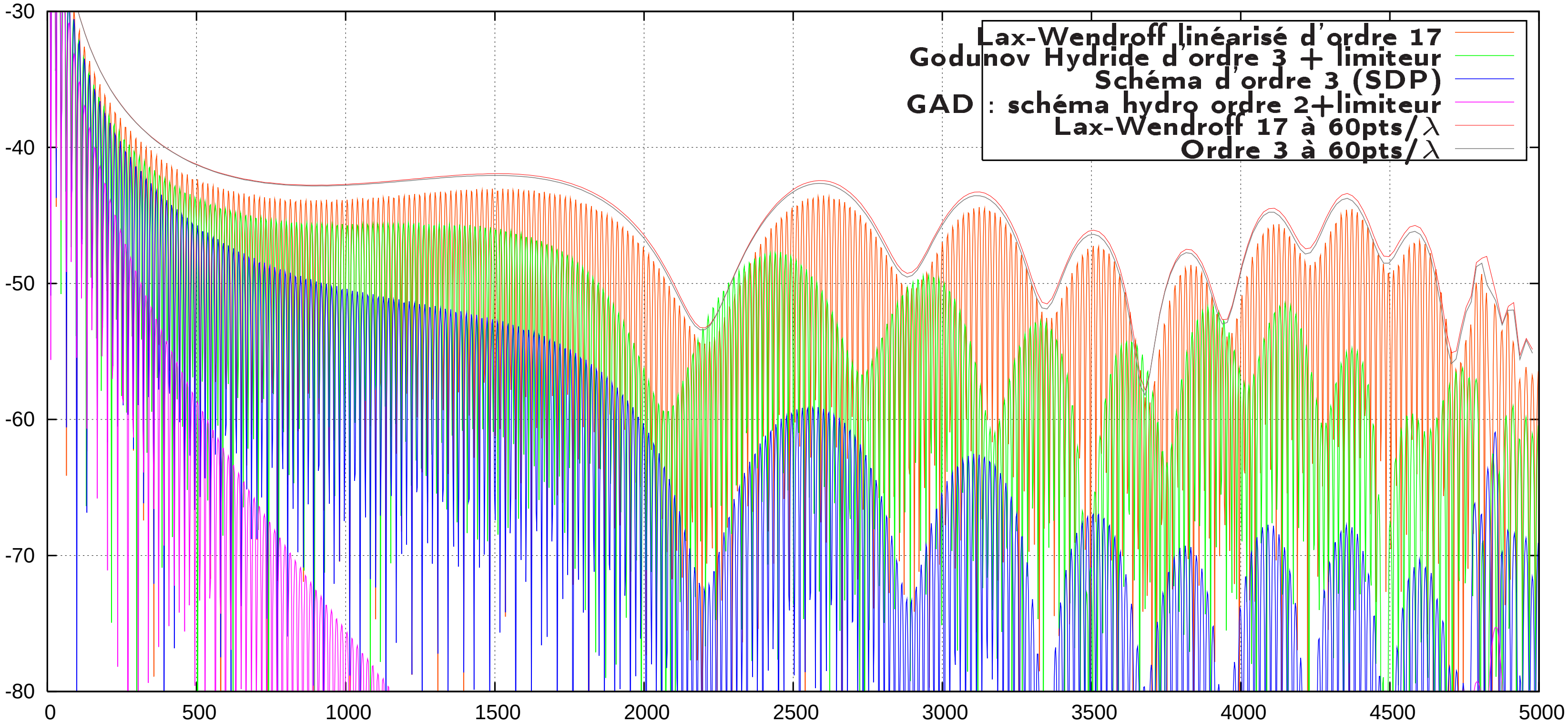
$c(y) = 343.23 + 0.1y$

$\rho = \rho_0 = 1.205$

$p_s(t) = sin(20\pi t)$

$p_{(t=0)} = 0$

$u_{(t=0)} = v_{(t=0)} = 0$



Atténuation (dB) de la pression suivant x à $y = 1m$

(15pts/λ, 4M mailles, < 9mn16CPUs, rectifié 3D, i.e. symétrie cylindrique).

The red curve if close to the reference solution (quasi-analytical solution). The attenuation for low order schemes makes then useless.

The CPU cost of the high order schemes : COST $= an_{mem} + bn_{flops}$
with $a_{\text{RAM}} = 0.1\mu s$, $b = 0.0003\mu s$: the ratio is $> 300!$

Schémas	# flops/maille	μs/maille	Mflops$\cdot s^{-1}$
Lax-Wendroff 17	1336	0.427	2690
Lax-Wendroff 9	432	0.259	1880
Lax-Wendroff 2	56	0.104	870
Lax-Wendroff 1	40	0.080	900
GAD	116	0.128	1100
Ordre 2	200	0.101	2100
Ordre 3	950	0.342	2400
Godunov (UpWind)	46	0.065	1100

Here high order costs not too much !!

For a 1D linear advection the ration of flops is $\approx \frac{1300}{4} \approx 200$.

Conclusion and pespectives

Lagrange+remapp schemes can be analyzed with entropic schemes.

These entropic schemes (S increases) are stable.

A general theorem states that almost all hyperbolic models of continuum mechanics can be incorporated with this approach.

I know about no direct eulerian code compatible with such an approach.

Current work

- axisymmetric MHD
- very high order methods.
- All these studies are AMR oriented.