Chapter 4

From kinetic to fluid: Hydrodynamic limits

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Summary

1. Macroscopic description of particle systems

Fluid variables

- Fluid quantities = averaged over a 'small' volume in physical space
- Ex. Density $n(x,t)\,dx =$ number of particles in a small volume dx.

$$\text{Mean momentum } q\,dx = \sum_{i \in dx} v_i$$

$$\text{Mean energy } W\,dx = \sum_{i \in dx} |v_i|^2/2$$

Link w. the kinetic distribution function

- $$\begin{pmatrix} n \\ q \\ 2W \end{pmatrix} = \int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv$$
- n, q, W, ... are moments of f
  - Eqs for n, q, W, ... are called fluid (or macroscopic) equations
  - Ex. Euler, Navier-Stokes, Drift-Diffusion, etc.

Hydrodynamic limits

- How to derive fluid eqs from kinetic eqs ?

$$\text{Particle} \xrightarrow{1} \text{Kinetic} \xrightarrow{2} \text{Fluid}$$

$$\xrightarrow{1} \begin{cases} \text{Mean-Field limit} \\ \text{Boltzmann-Grad limit} \end{cases}$$

$$\xrightarrow{2} \begin{cases} \text{Hydrodynamic limit} \\ \text{Diffusion limit} \end{cases}$$

2. The moment method

Moment method

- Natural idea: (i) multiply Boltzmann eq. by $1,\ v,\ |v|^2$ and integrate wrt v:

$$\int ((\partial_t + v \cdot \nabla_x) f - Q(f)) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv$$

- (ii) use conservations:

$$\int Q(f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

- (iii) Get conservation eqs

$$\frac{\partial}{\partial t}\begin{pmatrix} n \\ q \\ 2W \end{pmatrix} + \nabla_x \cdot \int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} v\, dv = 0$$

- Problem: Express fluxes in term of the conserved variables $n,\ q,\ W$
  - $\int f v_i v_j\, dv$ (for $i \neq j$) and $\int f |v|^2\, v\, dv$ cannot be expressed in terms of $n,\ q,\ W$.
- conservation eqs are not closed

Fluxes

- Density flux: $\int f v \, dv = q$. Define

$$u = \frac{q}{n} \text{ Velocity}$$

- Momentum flux tensor:

$$\begin{aligned} \int f v v \, dv &= \int f u u \, dv + \int f (v-u)(v-u) \, dv \\ &= n u u + \mathbb{P} \end{aligned}$$

$\mathbb{P}$ pressure tensor, not defined in terms of n, q, W

- Energy flux

$$\int f|v|^2 \, v \, dv = 2(Wu + \mathbb{P}u + \mathbb{Q}u)$$

$$2\mathbb{Q} = \int f|v-u|^2(v-u)\, dv$$

not defined in terms of n, q, W

Conservation equations

➠

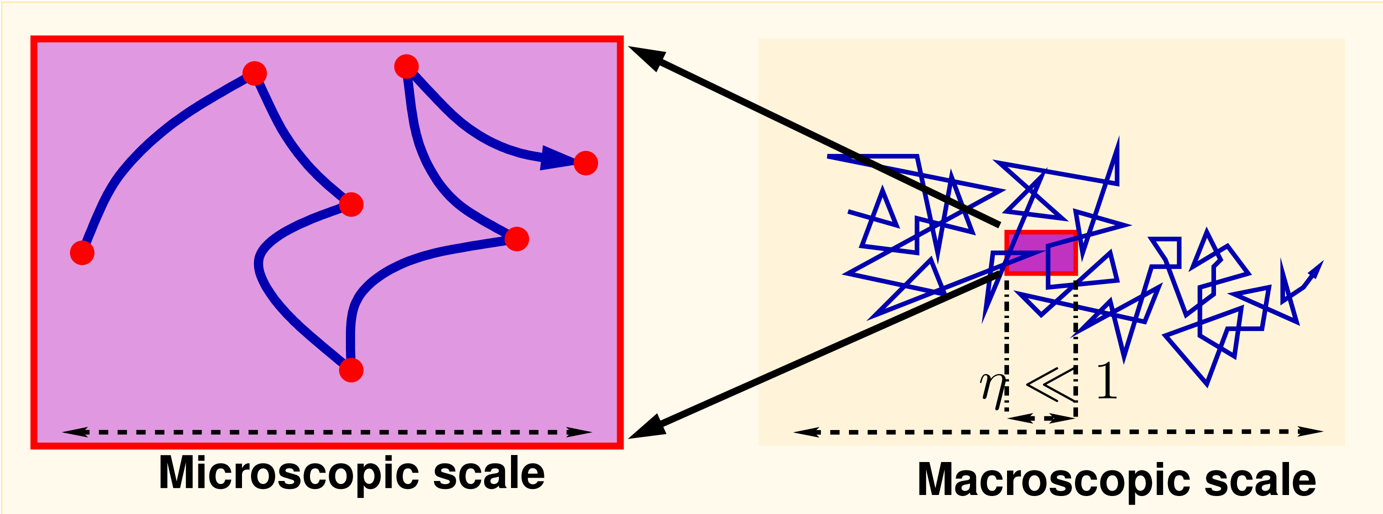
$$\frac{\partial}{\partial t}\begin{pmatrix} n \\ q \\ W \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} nu \\ nuu + \mathbb{P} \\ Wu + \mathbb{P}u + \mathbb{Q} \end{pmatrix} = 0$$

➠ Problem: find a prescription which relates $\mathbb{P}$ and $\mathbb{Q}$ to n, u, W:

Closure problem

3. Local thermodynamical equilibrium: Euler eq.

Hydrodynamic scaling



➠ Rescale: $x' = \varepsilon x$, $t' = \varepsilon t$

$$\varepsilon(\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon) = Q(f^\varepsilon)$$

Limit $\varepsilon \to 0$

- Suppose $f^\varepsilon \to f_0$ smoothly. Then

$$Q(f_0) = 0$$

i.e. $\exists n(x,t),\ u(x,t),\ T(x,t)$ s.t. $f = M_{n,u,T}$

-

$$\begin{pmatrix} n^\varepsilon \\ n^\varepsilon u^\varepsilon \\ 2W^\varepsilon \end{pmatrix} \to \begin{pmatrix} n \\ nu \\ 2W = n|u|^2 + 3nT \end{pmatrix}$$

Fluxes

- $$\mathbb{P}^{\varepsilon} = \int f^{\varepsilon} (v-u)(v-u)\, dv \longrightarrow \mathbb{P} = p\, \mathbf{Id}$$

$$p = nT = \text{ Pressure}$$

- $$2\mathbb{Q}^{\varepsilon} = \int f^{\varepsilon} |v-u|^2 (v-u)\, dv \longrightarrow 0$$

- $$\frac{\partial}{\partial t}\begin{pmatrix} n \\ nu \\ n|u|^2 + 3nT \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} nu \\ nuu + nT\,\mathrm{Id} \\ (n|u|^2 + 5nT)u \end{pmatrix} = 0$$

- Euler eqs of gas dynamics.
  $p = nT$ perfect gas Equation-of-State

4. Hilbert expansion and the Navier-Stokes eq.

Beyond Euler

- Problem: find order ε, $\varepsilon^2, \ldots$ corrections to Euler eqs.

- Expand (Hilbert expansion):

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$$

Insert in the Boltzmann eq.

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

Hilbert expansion

$$
\begin{aligned}
&-\varepsilon^{-1} Q(f_0) \\
&+\varepsilon^{0}((\partial_t + v \cdot \nabla_x) f_0 - L f_1) \\
&+\varepsilon^{1}((\partial_t + v \cdot \nabla_x) f_1 - (1/2) D(f_1, f_1) - L f_2) \\
&+ \ldots = 0
\end{aligned}
$$

$L f_1 = DQ(f_0) \cdot f_1$ First derivative

$D(f_1, f_1) = D^2 Q(f_0)(f_1, f_1)$ Second derivative

Linearized BGK operator

- Simplification: $Q =$ BGK operator

$$Q(f) = -\nu(f - M_f)$$

- Linearized BGK operator:

$$Lf_1 = -\nu(f_1 - \mathcal{M}_{f_1})$$
$$\mathcal{M}_{f_1} = (A + B \cdot v + C|v|^2)M_{f_0}$$

$A, C \in \mathbb{R}$, $\quad B \in \mathbb{R}^3$ uniquely determined by

$$\int (f_1 - \mathcal{M}_{f_1}) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

Linearized Maxwellian

- $\mathcal{M}_{f_1}$: Linearized Maxwellian about $M_{f_0} = M_{n,u,T}$.

- Alternate expression

$$\mathcal{M}_{n_1,u_1,T_1} = (\frac{n_1}{n} + \frac{v-u}{T} \cdot u_1 + (\frac{|v-u|^2}{2T^2} - \frac{3}{2T})T_1) M_{n,u,T}$$

where $n_1,\ u_1,\ T_1$ are given by

$$\begin{pmatrix} n_1 \\ nu_1 \\ 3nT_1 \end{pmatrix} = \int f_1 \begin{pmatrix} 1 \\ v-u \\ |v-u|^2 - 3T \end{pmatrix} dv = 0$$

Projection on the linearized Maxwellian

- $f \longrightarrow \mathcal{M}_f$ is a projector Π:

$$\Pi^2 = \Pi$$

- Linearized BGK operator:

$$Lf = -\nu(f - \Pi f)$$

satisfies $\Pi L = 0$.

- We also write $$\pi f = \int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv$$

$$\Pi f = 0 \Longleftrightarrow \pi f = 0$$

Properties of L

- (i) Null-Space:

$$Lf_1 = 0 \Longleftrightarrow \exists n_1,\ u_1,\ T_1 \text{ s.t. } f_1 = \mathcal{M}_{n_1,u_1,T_1}$$

- (ii) Collisional invariants

$$\int Lf_1\, g\, dv = 0 \Longleftrightarrow g = (A + B \cdot v + C|v|^2)$$

where $A, C \in \mathbb{R}\,, \quad B \in \mathbb{R}^3$ arbitrary

Properties of L (cont)

- (iii) pseudo-inverse: g_1 given.

$$\exists f_1 \text{ s.t. } L f_1 = g_1 \iff$$

$$\pi g_1 := \int g_1 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

$$f_1 = -\nu^{-1} g_1 + \mathcal{M}_{n_1, u_1, T_1} \quad n_1, u_1, T_1 \text{ arbitrary}$$

- $f_1 = -\nu^{-1} g_1$ uniquely characterized by

$$\pi f_1 = 0$$

Pseudo-inverse

Hilbert expansion (cont)

- Cancel each term of the expansion
- Order ε^{-1}: $\exists\, n,\ u,\ T$ s.t.

$$f_0 = M_{n,u,T}$$

- Order ε^0:

$$L f_1 = (\partial_t + v \cdot \nabla_x) f_0$$

First order perturbation equation. Solvability ?

First order perturbation eq.

- $\exists f_1 \iff$

$$\pi(\partial_t + v \cdot \nabla_x) f_0 := \int (\partial_t + v \cdot \nabla_x) f_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

$\iff$ n, u, T satisfy the Euler eq.

- Solve for f_1:

$$f_1 = -\nu^{-1}(\partial_t + v \cdot \nabla_x) f_0 + \mathcal{M}_{n_1, u_1, T_1}$$

n_1, u_1, T_1 arbitrary

Second order

- order ε^1:

$$Lf_2 = (\partial_t + v \cdot \nabla_x)f_1 - (1/2)D(f_1, f_1)$$

- Solvability cnd: $\exists f_2$

$$\iff \pi((\partial_t + v \cdot \nabla_x)f_1 - (1/2)D(f_1, f_1)) = 0$$

i.e.

$$-\pi(\partial_t + v \cdot \nabla_x)(\nu^{-1}(\partial_t + v \cdot \nabla_x)f_0)$$
$$+\pi(\partial_t + v \cdot \nabla_x)\mathcal{M}_1 - (1/2)\pi D(f_1, f_1) = 0$$

Details of 2nd order solvability eq.

- (i) Derivatives of Q at any order satisfy the conservation properties:

$$\pi D(f_1, f_1) = 0$$

- (ii) Denote $\mathcal{LE}(n_1, u_1, T_1)$ the linearized Euler operator about n, u, T acting on (n_1, u_1, T_1). Then:

$$\pi(\partial_t + v \cdot \nabla_x)\mathcal{M}_{n_1,u_1,T_1} = \mathcal{LE}(n_1, u_1, T_1)$$

Ex: linearized density conservation operator:

$$\mathcal{LE}(n_1, u_1, T_1)_1 = \partial_t n_1 + \nabla_x \cdot (n u_1 + n_1 u)$$

etc.

Derivatives of Maxwellian (1)

- (iii) Euler equations for $n, u, T \Longrightarrow$

$$\pi \partial_t (\nu^{-1} (\partial_t + v \cdot \nabla_x) f_0) = \partial_t \pi (\nu^{-1} (\partial_t + v \cdot \nabla_x) f_0) = 0$$

- (iv) Last term:

$$\pi (v \cdot \nabla_x) (\nu^{-1} (\partial_t + v \cdot \nabla_x) f_0) = \nabla_x \cdot (\pi v \nu^{-1} (\partial_t + v \cdot \nabla_x) f_0)$$

- requires the computation of:

$$(\partial_t + v \cdot \nabla_x) M_{n,u,T} = \frac{\partial M}{\partial (n, u, T)} (\partial_t + v \cdot \nabla_x) (n, u, T)^T$$

Derivatives of Maxwellian (2)

- Euler equation $\longrightarrow$ replace time derivatives of (n, u, T) by space derivatives

$$(\partial_t + v \cdot \nabla_x) M = (\mathcal{A} : \sigma(u) + \mathcal{B} \cdot \nabla T) M$$

- with

$$\mathcal{A} = \frac{1}{2}\left(\frac{(v-u)(v-u)}{T} - \frac{|v-u|^2}{3T}\mathrm{Id}\right)$$

$$\mathcal{B} = \left(\frac{|v-u|^2}{2T} - \frac{5}{2}\right)\frac{v-u}{T}$$

$$\sigma(u) = \nabla u + (\nabla u)^T - \frac{2}{3}(\nabla \cdot u)\mathrm{Id}$$

Properties of $\mathcal{A}$ and $\mathcal{B}$

- $\pi\mathcal{A} = 0$, $\pi\mathcal{B} = 0$
- computation of $\pi(v\mathcal{A})$ and $\pi(v\mathcal{B})$ (omitted) gives

$$\pi v \nu^{-1}(\partial_t + v \cdot \nabla_x) f_0 = \begin{pmatrix} 0 \\ \mu\sigma(u) \\ 2(\mu\sigma(u)u + \kappa\nabla T) \end{pmatrix}$$

$$\mu = \nu^{-1} n T = \text{ viscosity}$$
$$\kappa = (5/2)\nu^{-1} n T = \text{ heat conductivity}$$

Second order solvability cnd (summary)

$$\mathcal{L}\mathcal{E}(n_1, u_1, T_1) = \begin{pmatrix} 0 \\ \nabla_x(\mu\sigma(u)) \\ 2\nabla_x(\mu\sigma(u)u + \kappa\nabla T) \end{pmatrix}$$

Linearized Euler with rhs depending on second order derivatives of the leading order terms.

- Define

$$n^\varepsilon = n_0 + \varepsilon n_1,\ u^\varepsilon = u_0 + \varepsilon u_1,\ T^\varepsilon = T_0 + \varepsilon T_1$$

- Up to $O(\varepsilon^2)$ terms, n^ε, u^ε, T^ε satisfy the Navier-Stokes equations

$$\begin{aligned}
&\partial_t n + \nabla_x \cdot nu = 0 \\
&\partial_t nu + \nabla_x \cdot (nuu + nT\,\mathbf{Id}) = \varepsilon \nabla_x(\mu\sigma(u)) \\
&\partial_t(n|u|^2 + 3nT) + \nabla_x \cdot ((n|u|^2 + 5nT)u) = \\
&\qquad\qquad 2\varepsilon \nabla_x(\mu\sigma(u)u + \kappa \nabla T)
\end{aligned}$$

Remarks on Navier-Stokes eqs

- Diffusion terms of order $O(\varepsilon)$
- Diffusion terms of order $O(1)$, requires small velocities i.e. rescaling $u \to \varepsilon u$. Gives incompressible Navier-Stokes eq.
- Higher orders: $O(\varepsilon^2) \longrightarrow$ Burnett.
  contains 3rd order derivatives (dispersive) $\Longrightarrow$ ill-posed.
  Same for higher order ($O(\varepsilon^3) =$ Super-Burnett)
- Stationary sols of Euler $\neq$ stationary sols of NS (cf replacement of time derivatives by space derivatives). Same at higher orders.

The distribution function up to $O(\varepsilon^2)$

- Build the approximate solution:

$$f^\varepsilon = M_{n,u,T} + \varepsilon(\tilde{f}_1 + \mathcal{M}_{n_1,u_1,T_1}) + O(\varepsilon^2)$$

$$\tilde{f}_1 = -\nu^{-1}(\partial_t + v \cdot \nabla_x)M \,, \quad \pi \tilde{f}_1 = 0$$

- Note

$$M_{n^\varepsilon,u^\varepsilon,T^\varepsilon} = M_{n,u,T} + \varepsilon \mathcal{M}_{n_1,u_1,T_1} + O(\varepsilon^2)$$

Then

$$f^\varepsilon = M_{n^\varepsilon,u^\varepsilon,T^\varepsilon} + \varepsilon \tilde{f}_1 + O(\varepsilon^2)$$

$$\pi f^\varepsilon - \pi M_{n^\varepsilon,u^\varepsilon,T^\varepsilon} = O(\varepsilon^2)$$

Need for the Chapman-Enskog expansion

- f^ε and $M_{n^\varepsilon,u^\varepsilon,T^\varepsilon}$ have the same moments up to terms of order $O(\varepsilon^2)$
- Hilbert expansion does not produce $M_{n^\varepsilon,u^\varepsilon,T^\varepsilon}$ directly
- Can we modify Hilbert expansion in a such a way that $M_{n^\varepsilon,u^\varepsilon,T^\varepsilon}$ appears as the leading order term ?
- Chapman-Enskog expansion

5. Navier-Stokes eq. via the Chapman-Enskog expansion

Modified expansion

- $f^\varepsilon = f_0^\varepsilon + \varepsilon f_1^\varepsilon + \varepsilon^2 f_2^\varepsilon + \ldots$
  f_k^ε may depend on ε but is still formally $O(1)$.

- Leading order satisfies $Q(f_0^\varepsilon) = 0$. Implies

$$f_0^\varepsilon = M_{n^\varepsilon, u^\varepsilon, T^\varepsilon}$$

- We impose

$$\pi f_k^\varepsilon = 0, \; \forall k \geq 1$$

$$
\begin{aligned}
&+\varepsilon^0((\partial_t + v \cdot \nabla_x) f_0 - L f_1) \\
&+\varepsilon^1((\partial_t + v \cdot \nabla_x) f_1 - (1/2) D(f_1, f_1) - L f_2) \\
&+ \ldots = 0
\end{aligned}
$$

➠ Applying Π and using that $\Pi L = 0$:

$$\Pi(\partial_t + v \cdot \nabla_x) f_0 = O(\varepsilon)$$

$$
\begin{aligned}
&\varepsilon^0((\mathbf{Id} - \Pi)(\partial_t + v \cdot \nabla_x) f_0 - L f_1) \\
&+\varepsilon^1((\partial_t + v \cdot \nabla_x) f_1 + \varepsilon^{-1} \Pi(\partial_t + v \cdot \nabla_x) f_0 \\
&\qquad - (1/2) D(f_1, f_1) - L f_2) + \ldots = 0
\end{aligned}
$$

First order perturbation eq.

- Identify to 0 term by term:
- First order perturbation equation

$$Lf_1 = (\mathbf{Id} - \Pi)(\partial_t + v \cdot \nabla_x) f_0$$

Solvable by construction
We request $\Pi f_1 = 0$
Unique solution:

$$\begin{aligned} f_1 &= -\nu^{-1}(\mathbf{Id} - \Pi)(\partial_t + v \cdot \nabla_x) f_0 \\ &= -\nu^{-1}(\mathcal{A} : \sigma(u) + \mathcal{B} \cdot \nabla T) M \end{aligned}$$

second order perturbation eq.

$$Lf_2 = (\partial_t + v \cdot \nabla_x) f_1 + \varepsilon^{-1} \Pi (\partial_t + v \cdot \nabla_x) f_0$$
$$-(1/2) D(f_1, f_1)$$

➠ Solvability cnd

$$\pi(\partial_t + v \cdot \nabla_x) f_0 + \varepsilon \pi (\partial_t + v \cdot \nabla_x) f_1 = 0$$

$\pi(\partial_t + v \cdot \nabla_x) f_0 \to$ full Euler operator
$\pi \partial_t f_1 = \partial_t \Pi f_1 = 0$

$\pi(v \cdot \nabla_x) f_1$ already computed in the Hilbert expansion:

gives the Navier-Stokes terms

Solvability condition: result

- The solvability condition for f^2 directly gives the Navier-Stokes equation

$$
\begin{aligned}
&\partial_t n + \nabla_x \cdot nu = 0 \\
&\partial_t nu + \nabla_x \cdot (nuu + nT\,\mathbf{Id}) = \varepsilon \nabla_x (\mu \sigma(u)) \\
&\partial_t (n|u|^2 + 3nT) + \nabla_x \cdot ((n|u|^2 + 5nT)u) = \\
&\qquad\qquad 2\varepsilon \nabla_x (\mu \sigma(u) + \kappa \nabla T)
\end{aligned}
$$

6. Remarks and overview of rigorous results

Entropy for Euler

- H-theorem $\Longrightarrow$

$$\frac{\partial}{\partial t}\int f(\ln f - 1)dv + \nabla_x \cdot \int f(\ln f - 1)v\,dv \leq 0$$

- Euler: $f^\varepsilon \xrightarrow{\varepsilon\to 0} M_{n,u,T}$. Specific entropy S:

$$nS = \int M(\ln M - 1)dv = n\left(\ln\frac{n}{(2\pi T)^{3/2}} - \frac{5}{2}\right)$$

Entropy inequality for Euler ($=$ for smooth, $<$ for weak):

$$\frac{\partial}{\partial t}(nS) + \nabla_x \cdot (nSu) \leq 0$$

Entropy for Navier-Stokes

$$\frac{\partial}{\partial t}(nS) + \nabla_x \cdot \left(nSu + \varepsilon\kappa \frac{\nabla_x T}{T} \right) = -\varepsilon \left(\frac{\mu}{T} \sigma(u) : \sigma(u) + \frac{|\nabla_x T|^2}{T^2} \right) \leq 0$$

- Burnett or super-Burnett not consistent with the entropy inequality

- (i) Boltzmann $\to$ compressible Euler

Theorem [Caflish, CPAM 1980] n, u, T smooth solutions of Euler on a time interval $[0, t^*]$ ($t^* <$ blow-up time of regularity), with initial data n_0, u_0, T_0.
$\exists \varepsilon_0 > 0$, $\forall \varepsilon < \varepsilon_0$, $\exists f^\varepsilon$ a solution of the Boltzmann equation with initial data M_{n_0,u_0,T_0} on $[0, t^*]$ and

$$\sup_{[0,t^*]} \| f^\varepsilon(t) - M_{n,u,T}(t) \| \leq C\varepsilon$$

Rigorous results for the hydrodynamic limit (2)

- Boltzmann $\rightarrow$ incompressible Navier-Stokes
- Perturbation of a global Maxwellian with $u = 0$.
  - Rescale velocity and time (diffusion limit)
  - ref: [De Masi, Esposito, Lebowitz], [Bardos, Golse, Levermore], [Bardos, Ukai], [Golse, Saint-Raymond]

More about hydrodynamic limits (1)

- Boundary layers:
  - Slip boundary conditions for the Navier-Stokes equation
  - ref. [Sone, Aoki], [Golse, Coron, Sulem]
- Choc profiles:
  - stationary solution of Boltzmann equation which connects states at infinty connected with the Rankine-Hugoniot relation
  - ref. [Caflish, Nicolaenko], [Bardos, Golse, Nicolaenko]

More about hydrodynamic limits (2)

- Kinetic schemes:
  - use of kinetic eqs to derive schemes for the Euler eq.
  - ref. [Pullin], [Deshpande], [Perthame], [Lions, Tadmor, Perthame], [Bouchut]
- Fluid-kinetic coupling
  - through boundary layer analysis and kinetic schemes
  - ref. [Struckmeier et al], [Le Tallec et al]

More about hydrodynamic limits (3)

- Relaxation systems
  - Similar structure
  - use of relaxation to 'stabilize' Burnett equations via Chapman-Enskog like expansions
  - ref. [Chen, Liu, Levermore], [Jin, Xin], [Jin, Slemrod]
- Asymptotic preserving schemes
  - Schemes for the kinetic equation which are valid in the hydrodynamic limit
  - ref. [Klar], [Jin, Pareschi, Russo]

Criticism of Navier-Stokes

- When ∇_x large (transition regime)
  - Correction terms not small
  - Perturbation approach not valid

- Example of flaw:

$$
\begin{aligned}
f^\varepsilon &= M_{n^\varepsilon, u^\varepsilon, T^\varepsilon} + \varepsilon \tilde{f}_1 + O(\varepsilon^2) \\
&= M^\varepsilon - \varepsilon \nu^{-1} (\mathcal{A} : \sigma(u) + \mathcal{B} \cdot \nabla T) M^\varepsilon + O(\varepsilon^2)
\end{aligned}
$$

May be non-positive

Loss of realizability

Cures of failures of Navier-Stokes (1)

- First idea: Go beyond Navier-Stokes in the Hilbert (or Chapman-Enskog) expansion:
  $\longrightarrow$ not good: if first order perturbation not small, higher order ones will not be either !

- Example: Burnett not consistent with entropy dissipation

Cures of failures of Navier-Stokes (2)

- Second idea: Try to increase the number of moments
  - Moment system hierarchies
  - ref. [Grad], [Muller, Ruggeri (extended thermodynamics)], [Levermore]
- Try to do it consistently with the entropy dissipation rule
  - Levermore models (see applications in [Anile, Russo et al])
  - Developped in the next section

7. Higher order moment systems: Levermore's approach

Moments (1)

- List of monomials $\mu_i(v)$

$$\mu(v) = (\mu_i(v))_{i=0}^{N}$$

- Contains hydrodynamic moments

$$\mu_0(v) = 1;\ \mu_i(v) = v_i,\ i = 1, 2, 3;\ \mu_4(v) = |v|^2$$

- Example

$$\begin{aligned} \mu(v) &= \{1, v, vv\} \text{ Gaussian model} \\ \mu(v) &= \{1, v, vv, |v|^2 v, |v|^4\} \end{aligned}$$

- For a distribution function f, define:

$$m(f) = (m_i(f))_{i=0}^{N}\,, \quad m_i(f) = \int f\mu_i(v)\,dv$$

- Eq. for the i-th moment:

$$\frac{\partial}{\partial t} m_i(f) + \nabla_x \cdot \int f\mu_i(v) v\,dv = \int Q(f)\mu_i(v)\,dv$$

- Note $\int Q(f)\mu_i(v)\,dv \neq 0$ if $\mu_i \neq$ hydrodynamic monomial

Closure problem

- Find a prescription for

$$\int f \mu_i(v) v \, dv \text{ and } \int Q(f) \mu_i(v) \, dv$$

in terms of the moments m_i

Entropy minimization principle (Gibbs)

- Let $n, T \in \mathbb{R}_+$, $u \in \mathbb{R}^3$ fixed.

$$\min\{H(f) = \int f(\ln f - 1)dv \text{ s.t.}$$

$$\int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} n \\ nu \\ n|u|^2 + 3nT \end{pmatrix}\}$$

is realized by $f = M_{n,u,T}$.

Proof of Gibbs principle

- Euler-Lagrange eqs of the minimization problem:
  $\exists A, C \in \mathbb{R},\ B \in \mathbb{R}^3$ (Lagrange multipliers) s.t.

$$\int (\ln f - (A + B \cdot v + C|v|^2))\, \delta f\, dv = 0,\ \forall\, \delta f$$

- $\Longrightarrow f = \exp(A + B \cdot v + C|v|^2)$
  i.e. $f =$ Maxwellian

Euler eqs in view of the entropy principle

- Euler eqs = moment system (only involving hydrodynamical moments), closed by a solution of the entropy minimization principle

- Idea [Levermore], [extended thermodynamics]
Use the same principle for higher order moment systems

- Given a set of moments $m = (m_i)_{i=0}^N$, solve

$$\min\{H(f) = \int f(\ln f - 1) dv \text{ s.t. } \int f \mu(v) dv = m\}$$

- Solution: generalized Maxwellian:
  $\exists$ vector $\alpha = (\alpha_i)_{i=0}^N$ s.t.

$$f = M_\alpha(v) = \exp(\alpha \cdot \mu(v)) = \exp(\sum_{i=0}^N \alpha_i \mu_i(v))$$

Levermore moment systems

- Use the generalized Maxwellian M_α as a prescription for the closure

$$\frac{\partial}{\partial t}\int M_\alpha \mu(v)\, dv + \nabla_x \cdot \int M_\alpha \mu(v) v\, dv = \int Q(M_\alpha)\mu(v)\, dv$$

Gives an evolution system for the parameter α

➠ Has the form of a symmetrizable hyperbolic system: Define

$$\Sigma(\alpha) = \int M_\alpha \, dv = \int \exp(\alpha \cdot \mu(v)) \, dv$$

$$\phi(\alpha) = \int M_\alpha v \, dv = \int \exp(\alpha \cdot \mu(v)) v \, dv$$

$\Sigma(\alpha) =$ Massieu-Planck potential, $\phi =$ flux potential

$$\frac{\partial \Sigma}{\partial \alpha} = \int M_\alpha \mu(v) \, dv \,, \qquad \frac{\partial \phi}{\partial \alpha} = \int M_\alpha \mu(v) v \, dv$$

Symmetrized form

- Moment system $\equiv$

$$\frac{\partial}{\partial t}\frac{\partial \Sigma}{\partial \alpha} + \nabla_x \cdot \frac{\partial \phi}{\partial \alpha} = r(\alpha)$$

$$r(\alpha) = \int Q(M_\alpha)\mu(v)\, dv$$

- or

$$\frac{\partial^2 \Sigma}{\partial \alpha^2}\frac{\partial \alpha}{\partial t} + \frac{\partial^2 \phi}{\partial \alpha^2} \cdot \nabla_x \alpha = r(\alpha)$$

$\partial^2\Sigma/\partial\alpha^2 = \int M_\alpha \mu(v)\mu(v)\, dv$ symmetric $\gg 0$
$\partial^2\phi/\partial\alpha^2 = \int M_\alpha \mu(v)\mu(v) v\, dv$ symmetric

Hyperbolicity

- Hyperbolicity $\longrightarrow$ well posedness (Godounov, Friedrichs)
- $\neq$ Grad systems: not everywhere locally well-posed

- $S(m)$ = Legendre dual of $\Sigma(\alpha)$:

$$S(m) = \alpha \cdot m - \Sigma(\alpha)$$

where α is such that

$$m = \frac{\partial \Sigma}{\partial \alpha} (= \int M_\alpha \mu(v)\, dv)$$

- Then

$$\alpha = \frac{\partial S}{\partial m}$$

Entropy (cont)

- α and m are conjugate variables.
  - $\alpha =$ entropic (or intensive) variables
  - $m =$ conservative (or extensive) variables
- Link with H

$$S(m) = \int (\alpha \cdot \mu - 1) M_\alpha \, dv$$
$$= \int (\ln M_\alpha - 1) M_\alpha \, dv = H(M_\alpha)$$

Fluid entropy = Kinetic entropy evaluated at equilibrium

$$\partial_t m + \nabla_x \cdot \frac{\partial \phi}{\partial \alpha}\left(\frac{\partial S}{\partial m}(m)\right) = r\left(\frac{\partial S}{\partial m}(m)\right)$$

- Entropy inequality

$$\partial_t S(m) + \nabla_x \cdot F(m) = \frac{\partial S}{\partial m} \cdot r$$

$$F(m) = \alpha \cdot \frac{\partial \phi}{\partial \alpha} - \phi(\alpha) = \text{ Entropy flux}$$

with $\alpha = \partial S / \partial m$

Entropy dissipation

$$\frac{\partial S}{\partial m} \cdot r = \alpha \cdot \int Q(M_\alpha)\mu \, dv$$
$$= \int Q(M_\alpha) \ln M_\alpha \, dv \leq 0$$

Thanks to H-theorem

- Levermore system compatible with the entropy dissipation

$$\partial_t S(m) + \nabla_x \cdot F(m) \leq 0$$

Entropy dissipation $= 0$ iff $M_\alpha =$ standard Maxwellian $M_{n,u,T}$

Example: Gaussian closure

- $\mu(v) = \{1, v, vv\}$.

$$M_\alpha = \frac{n}{(\det 2\pi\Theta)^{1/2}} \exp\left(-\frac{1}{2}(v-u)\Theta^{-1}(v-u)\right)$$

Θ symmetric $\gg 0$ matrix
$\alpha \sim (n, u, \Theta)$

$$\partial_t n + \nabla_x \cdot nu = 0$$

$$\partial_t nu + \nabla_x \cdot (nuu + n\Theta) = 0$$

$$\partial_t (nuu + n\Theta) + \nabla_x \cdot (nuuu + 3n\Theta \wedge u) = Q(n, \Theta)$$

Entropy in the Gaussian model

- Collisions

$$Q(n, \Theta) = \int Q(M_\alpha) v v \, dv$$

- Entropy: $S = n\sigma$
  Entropy flux: $F = n\sigma u$

$$\sigma = \ln \left(\frac{n}{(\det 2\pi\Theta)^{1/2}} \right) - \frac{5}{2}$$

General models: constraints

- If highest degree monomial of odd parity, integrals like $\int \exp(\alpha \cdot \mu)\mu \, dv$ diverge
  - Constraint on μ: The set of α s.t. the integrals converge has non-empty interior
  - Highest degree monomial must have even parity
- Moment realizability:
  - characterize the set of m such that $\exists \alpha$ and $m = \int \exp(\alpha \cdot \mu)\mu \, dv$
  - ref. [Junk], [Schneider]

Example: 5 moment model (in 1D)

- ref. [Junk]:
  - Moment realizability domain not convex
  - fluid Maxwellians lie at the boundary of the realizability domain
  - Fluxes and characteristic velocities $\longrightarrow \infty$ when $m \to$ Maxwell.
- Severe drawback since collision operators relax to Maxwellians

Problems (cont)

- Explicit formulae for $\int \exp(\alpha \cdot \mu)\mu \, dv$ and $\int \exp(\alpha \cdot \mu)\mu v \, dv$ not available beyond Gaussian model

- Inversion of $\alpha \to m$ not explicit. Iterative algorithms to solve the Legendre transform.

- Collision operator: $r(\alpha) = \int Q(M_\alpha)\mu \, dv$ does not give the right Chapman-Enskog limit. (viscosity and heat conductivity $<$ Navier-Stokes)
  - Needs to correct the collision operator [Levermore, Schneider].

Practical use of Levermore's moment models

- Successful applications in a selected number of cases
  - Gaussian model [Levemore, Morokoff]
  - P^2 model of radiative transfer [Dubroca]
- Give a systematic methodology to imagine new models and new closures.

8. Summary, conclusion and perspectives

Summary

- Kinetic $\longrightarrow$ fluid by the moment method
  - closure problem
  - Relaxation to equilibrium $\longrightarrow$ Euler
  - Correction to Euler (via the Hilbert or Chapman-Enskog expansion): $\longrightarrow$ Navier-Stokes
- Transition regimes: gradients are too large and Navier-Stokes breaks down
  - Need for new models

Summary (cont) and perspectives

- Levermore's attempt:
  - closure by means of the entropy minimization principle
  - Nice features (hyperbolicity) but some flaws (moment realizability)
- The 'race' to models for transition regime is still not won
  - Major challenge for kinetic theory in the future