Chapter 3.

Theory of the Boltzmann equation

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Summary

1. Properties of the Boltzmann collision operator
2. Overview of existence results
3. Variants of the Boltzmann equation
4. Summary and conclusion

1. Properties of the Boltzmann collision operator

The Boltzmann equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f)$$

$$Q(f) = \int_{v_1 \in \mathbb{R}^3} \int_{\vec{n} \in \mathbb{S}^2_+} \sigma(|v - v_1|, cos\theta)\, |v - v_1|$$

$$[f(v')f(v'_1) - f(v)f(v_1)]\ dv_1\, d\vec{n}$$

$$v' = v - (v - v_1, \vec{n})\vec{n}\,, \quad v'_1 = v_1 + (v - v_1, \vec{n})\vec{n}$$

$$\cos\theta = \frac{|(v - v_1, \vec{n})|}{|v - v_1|}$$

Microreversibility

- $\vec{n}$ fixed in $\mathbb{S}^2_+$:

$$(v, v_1) \xrightarrow{J} (v', v_1') \ \text{ involution}$$

$$J^2 = \mathbf{Id}\,, \quad J = J^{-1}\,, \quad \det J = 1$$

- $$\sigma(v, v_1) = \sigma(v', v_1')$$

  Microreversibility: Probability of the direct collision is the same as the inverse one

Weak form

- Let ψ be any "regular" test function
- Denote $f' = f(v')$, $f'_1 = f(v'_1)$, etc, $V_1 = v - v_1$.

$$\int Q(f)\psi dv =$$

$$= -\frac{1}{2}\int (f'\,f'_1 - f\,f_1)(\psi' - \psi)\sigma|V_1|d\vec{n}\,dv\,dv_1 =$$

$$-\frac{1}{4}\int (f'\,f'_1 - f\,f_1)(\psi' + \psi'_1 - \psi - \psi_1)\sigma|V_1|d\vec{n}\,dv\,dv_1$$

$$= \frac{1}{2}\int f\,f_1(\psi' + \psi'_1 - \psi - \psi_1)\sigma|V_1|d\vec{n}\,dv\,dv_1$$

Collisional invariant

- ψ collisional invariant $\Longleftrightarrow$

$$\int Q(f)\psi dv = 0\,, \quad \forall f$$

- $\Longleftrightarrow$ ψ satisfies

$$\psi(v') + \psi(v_1') - \psi(v) - \psi(v_1)$$

$\forall (v, v_1, v', v_1')$ s.t. $\exists \vec{n}$ and $(v', v_1') = J(v, v_1)$

- $\Longleftrightarrow \exists A, C \in \mathbb{R},\ B \in \mathbb{R}^3$ s.t.

$$\psi(v) = A + B \cdot v + C|v|^2$$

Conservations

- Collisional invariants $\Longrightarrow$

$$\int Q(f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

- $\Longleftrightarrow$ conservation of $\left\{ \begin{array}{l} \text{mass} \\ \text{momentum} \\ \text{energy} \end{array} \right.$

Conservations (cont)

$$\frac{\partial f}{\partial t} = Q(f) \Longrightarrow \frac{\partial f}{\partial t} \int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

H-theorem

- Take $\psi = \ln f$ in the weak formulation
- Use that $\ln$ is an increasing function

$$\int Q(f) \ln f \, dv =$$

$$-\frac{1}{4} \int (f' \, f'_1 - f \, f_1)(\ln f' + \ln f'_1 - \ln f - \ln f_1)$$

$$\sigma |V_1| d\vec{n} \, dv \, dv_1$$

$$= -\frac{1}{4} \int (f' \, f'_1 - f \, f_1)(\ln f' f'_1 - \ln f \, f_1) \sigma |V_1| d\vec{n} \, dv \, dv_1$$

$$\leq 0$$

- $$H(f) = \int f(\ln f - 1)dv$$

  Note $h(s) = s(\ln s - 1) \Longrightarrow h'(s) = \ln f$

- $$\frac{\partial f}{\partial t} = Q(f) \Longrightarrow \frac{\partial H(f)}{\partial t} = \int Q(f) \ln f dv \leq 0$$

  Entropy decay
  Rate of entropy decay = entropy dissipation

Irreversibility

Equilibria

$$Q(f) = 0 \Longrightarrow \int Q(f) \ln f \, dv = 0$$

$$\Longleftrightarrow \int (f' f_1' - f f_1)(\ln f' f_1' - \ln f f_1) \sigma |V_1| d\vec{n} \, dv \, dv_1 = 0$$

$\Longleftrightarrow \ln f$ is a collisional invariant

$\Longleftrightarrow \exists A, C \in \mathbb{R}_+, \; B \in \mathbb{R}^3$ s.t.

$$f = \exp(A + B \cdot v + C|v|^2)$$

➠ Maxwellian distribution

- Other expression:

$$M_{n,u,T} = \frac{n}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-u|^2}{2T} \right)$$

(n, u, T) straightforwardly related w. (A, B, C)

$$\int M_{n,u,T} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} n \\ nu \\ n|u|^2 + 3nT \end{pmatrix}$$

Characterization of Maxwellians

- (i) Entropy dissipation $\int Q(f) \ln f dv \leq 0$ and $\equiv 0$ iff $f =$ Maxwellian

- (ii) Entropy minimization subject to moment constraints: let $n, T \in \mathbb{R}_+$, $u \in \mathbb{R}^3$ fixed.

$$\min\{H(f) = \int f(\ln f - 1) dv \text{ s.t.}$$

$$\int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} n \\ nu \\ n|u|^2 + 3nT \end{pmatrix} \}$$

is realized by $f = M_{n,u,T}$.

2. Overview of existence results

Homogeneous equation

- $$\frac{\partial f}{\partial t} = Q(f)$$
- Existence and uniqueness of classical solutions [Carleman], [Arkeryd], ...
- Convergence to a Maxwellian as $t \to \infty$ [Desvillettes], [Wennberg], ...

Non-homogeneous equation

- $$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f)$$
- Difficulty: $Q(f)$ quadratic in f
- ref. [DiPerna, Lions]: renormalized solutions i.e. satisfying:

  $$\left(\frac{\partial}{\partial t} + v \cdot \nabla_x\right)\beta(f) = \beta'(f)Q(f) \text{ in } \mathcal{D}'$$

  $\forall \beta$ Lipschitz, s.t. $|\beta'(f)| \leq C/(1+f)$
- Note: $\beta'(f)Q(f)$ grows linearly with f

Perturbation of equilibria

- ref: [Ukai], [Nishida, Imai], ...
- M global Maxwellian (parameters (n, u, T) are constant indep. of x, t
- $f = M + g$, with "$g \ll M$"
- Decompose

$$Q(f) = L_M g + \Gamma(g, g)$$

- Prove operator $v \cdot \nabla_x g - L_M g$ dissipative
- Compensates blow-up of $\Gamma(g, g)$ if g small

3. Variants of the Boltzmann equation

BGK operator

$$Q(f) = -\nu(f - M_f)$$

where $M_f = M_{n,u,T}$ is the Maxwellian with the same moments as f i.e. (n, u, T) are such that

$$\int (M_f - f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

i.e.

$$\begin{pmatrix} n \\ nu \\ n|u|^2 + 3nT \end{pmatrix} = \int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv$$

Properties of BGK operator

- Shows the same 'algebraic' properties as the Boltzmann operator
- (i) Collisional invariants:

$$\int Q(f)\psi dv = 0, \forall f \iff \psi(v) = A + B \cdot v + C|v|^2$$

- (ii) Equilibria:

$$Q(f) = 0 \iff f = M_{n,u,T}$$

Properties of BGK operator (cont)

- H-theorem

$$\int Q(f) \ln f dv \le 0 \qquad (= 0 \Longleftrightarrow f = M_{n,u,T})$$

- Simpler operator
  - Theory simpler
  - Numerical simulations are easier
  - Some unphysical features (Prandtl number)

Some references for BGK

- Existence of weak solutions [Perthame, Pulvirenti]
- Numerical solutions [Dubroca, Mieussens]
- Generalized BGK models [Bouchut, Berthelin]

Landau operator

- grazing collision limit [Desvillettes]
- (i) Suppose $\exists$ parameter η s.t.

$$\sigma^\eta(|v - v_1|, \cos\theta) \text{"} \longrightarrow \text{"} \bar\sigma(|v - v_1|)\delta(\theta - \pi/2)$$

- $\Longrightarrow Q_B(f) \to Q_L(f)$

$$Q_L(f) = \nabla_v \cdot \int \bar\sigma(|v - v_1|) S(v - v_1) \\ (f_1 \nabla_v f - f (\nabla_v f)_1)\, dv_1$$

$$S(v) = \mathrm{\bf Id} - \frac{vv}{|v|^2}$$

➠ ref. [D., Lucquin]

$$Q_B^\eta(f) = \int\int_{|\theta-\pi/2|>\eta} \sigma_C(|v-v_1|, cos\theta)\,|v-v_1|$$
$$[f(v')f(v_1') - f(v)f(v_1)]\; dv_1\, d\vec{n}$$

with σ_C Coulomb scattering cross section. Note, Q_B^0 not defined because integral diverges

$$Q_B^\eta(f) \overset{\eta\to 0}{\sim} |\ln\eta| Q_L(f) + O(\eta)$$

$\ln\eta$: Coulomb logarithm

Theory of Landau equation

- Existence theory for Landau equation far from being as complete as for the Boltzmann equation
- Weak solutions for homogeneous equation [Arseneev]
- Linearized Landau equation [D., Lemou]
- Nonlinear Landau: Considerable amount of work recently by [Alexandre, Desvillettes, Villani, ...]

Other variants of Boltzmann

- Enskog eq. Keep the diameter of the spheres δ finite $\longrightarrow$ space delocalization of the operator.
  - Note: a result of convergence of a stochastic particle system to the Enskog eq. by [Rezhakanlou]

- Quantum Boltzmann eq.:
  $f f_1 \longrightarrow f f_1 (1 \pm f')(1 \pm f_1')$
  - $-$ sign: Pauli operator [Golse & Poupaud], [Dolbeault]
  - $+$ sign: Bose-Einstein operator [Mischler et al]

Variants (cont)

- Boltzmann for molecules with internal degrees of freedom $\longrightarrow$ ”real gases” (as opposed to ”perfect gases”
  - ref. [Neunzert, Strückmeier et al], [Le Tallec, Perthame et al], …

4. Summary and conclusion

Summary

- Properties of the Boltzmann operator
  - Conservation (collisional invariants)
  - Equilibria (Maxwellians)
  - Relaxation (entropy decay)
- Existence theory
  - Classical theory (perturbation of equilibria)
  - Renormalized solutions [DiPerna, Lions]
- Variants of Boltzmann
  - Model BGK operator
  - Grazing limit and the Landau operator

Equilibria and Hydrodynamic limits

- Properties of the Boltzmann operator $\longrightarrow$ derivation of hydrodynamic equations
- Use of BGK operator $\longrightarrow$ simpler theory