Learning and convergence analysis in finite mean field games

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Mean Field Games (MFGs) were introduced by Lasry, Lions (2007) and, independently, by Huang, Caines and Malhamé (2007). One of the main purposes of the theory is to develop a notion of Nash equilibria for dynamic games, which can be deterministic or stochastic, with an infinite number of players. Most of the literature about MFGs deals with games in continuous time and where the agents are distributed on a continuum of states. In this article we consider a MFG problem where the number of states and times are finite. For the sake of simplicity, we will call *finite MFGs* the games of this type. This framework has been introduced by Gomes, Mohr and Souza in [1], where the authors prove results related to the existence and uniqueness of equilibria, as well as the convergence to a stationary equilibrium as time goes to infinity. Our contributions to finite MFGs will be twofold:

1. we study an iterative method, similar to *fictitious play*, and prove its convergence to an equilibrium,
2. we investigate the relation between continuous and finite MFGs.

Let us give a brief definition of the model and our contributions on it. Let $\mathcal{S}$ be a finite set and $\mathcal{T}$ consists of $0 = t_0 < t_1 < \ldots < t_m = T$, representing the set of states and time. We denote by $\mathcal{P}(\mathcal{S})$ the set of probability measures over $\mathcal{S}$. We call a tuple (U, M) with $U : \mathcal{T} \times \mathcal{S} \to \mathbb{R}$, $M : \mathcal{T} \to \mathcal{P}(\mathcal{S})$, an equilibrium solution to the finite MFG, if there exists $\hat{P} : \mathcal{S} \times \mathcal{S} \times \mathcal{T} \setminus \{T\} \to [0, 1]$ such that:

$$\begin{aligned} (i) \quad & U(x, t_k) = \inf_{p \in \mathcal{P}(\mathcal{S})} \sum_{y \in \mathcal{S}} p_y \Big(c_{xy}(p, M(t_k)) + U(y, t_{k+1}) \Big), \\ (ii) \quad & \hat{P}(x, \cdot, t_k) \in \arg \min_{p \in \mathcal{P}(\mathcal{S})} \sum_{y \in \mathcal{S}} p_y \big(c_{xy}(p, M(t_k)) + U(y, t_{k+1}) \big), \\ (iii) \quad & M(x, t_{k+1}) = \sum_{y \in \mathcal{S}} M(y, t_k) \hat{P}(y, x, t_k), \end{aligned} \tag{1}$$

where $U(\cdot, T)$ and $M(0)$ are given. We should note that the relation (i)(1), is a discrete dynamic programming with a *backward* nature, while (1)(iii) describes the evolution of the initial measure M_0 with a *forward* equation. So due to this contrast, and the fact that the system is coupled, finding an equilibrium will be subtle. For achieving this goal, we apply an iterative scheme similar to the fictitious play procedure defined by Brown (1951) and construct $\{(U^n, M^n, \bar{M}^n)\}_{n \in \mathbb{N}}$ in the following recursive way. Let U^{n+1} be obtained from (i)(1) by putting $M = \bar{M}^n$. Denote P^n the corresponding minimiser of problem (ii)(1) for $M = \bar{M}^n$. Then, we obtain M^{n+1} from (iii)(1) by setting $\hat{P} = P^n$. At last set $\bar{M}^{n+1} = \frac{1}{n+1} M^{n+1} + \frac{n}{n+1} \bar{M}^n$. We proved the convergence of (U^n, M^n) to equilibrium, under monotonicity and suitable continuity assumptions. Our second contribution is the convergence of the solutions of finite MFG system (1) to the solution of first-order MFG system when discretization becomes finer. We answered to this question for the case of

$$c_{xy}(p, M) := \Delta t_n \left(L \left(x, \frac{y - x}{\Delta t_n} \right) + f(x, M) \right) + \epsilon_n \log(p_y),$$

and suitable assumptions on the rate of convergence of $\epsilon_n, \Delta t_n, \Delta x_n$ to zero. Numerical results and simulations are done by the help of the developed iterative method.

Références

[1] D. A. Gomes, J. Mohr, R. R. Souza *Discrete time, finite state space mean field games*, Journal de mathmatiques pures et appliques, 93(3), 308-328.

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[3] S. Hadikhanloo *Learning in anonymous nonatomic games with applications to first-order mean field games*, arXiv preprint arXiv:1704.00378.

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