High-order local time-stepping for Discontinuous Galerkin methods

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The Discontinuous Galerkin (DG) method is a general finite element method for approximating systems of conservation laws of the form

$$\partial_t \mathbf{w} + \sum_{k=1}^{D} \partial_k \mathbf{f}^k(\mathbf{w}) = \mathbf{g}. \tag{1}$$

The unknown is the vector of conservative variables $\mathbf{w}(\mathbf{x}, t) \in \mathbb{R}^m$ depending on the space variable $\mathbf{x} = (x^1, \ldots, x^D) \in \mathbb{R}^D$ and on time t. The source term $\mathbf{g}(x, t)$ is given.
Let $\mathbf{n} = (n_1, \ldots, n_D) \in \mathbb{R}^D$ be a spatial direction, the flux in direction $\mathbf{n}$ is defined by

$$\mathbf{f}(\mathbf{w}) \cdot \mathbf{n} = \sum_{k=1}^{D} n_k \mathbf{f}^k(\mathbf{w}).$$

For instance, in this work we consider the numerical simulation of an electromagnetic wave. In this particular case, the conservative variables are

$$\mathbf{w} = (\mathbf{E}^T, \mathbf{H}^T)^T \in \mathbb{R}^m, \quad m = 6.$$

where $\mathbf{E} \in \mathbb{R}^3$ is the electric field, $\mathbf{H} \in \mathbb{R}^3$ is the magnetic field. The flux is given by

$$\mathbf{f}(\mathbf{w}) \cdot \mathbf{n} = \begin{pmatrix} -\mathbf{n} \times \mathbf{H} \\ \mathbf{n} \times \mathbf{E} \end{pmatrix}.$$

The Discontinuous Galerkin (DG, see [2], for instance) is a general high order numerical method for solving system of the form (1).
The time integration is often done in a explicit way. Several approaches have been proposed to achieve high order in time (see for instance [1]). The scheme is then constrained by a CFL stability condition, which can be very costly for unstructured meshes with very different cell sizes.
We construct a new low-storage and high-order time integrator for the system of conservation laws (1). The method is based on methods of geometric numerical integration [3] and a graph colouring. It can handle meshes with large variations of cell sizes thanks to local time stepping. It is well adapted to Discontinuous Galerkin (DG) approximations. Several numerical experiments are presented for Maxwell's equations.

Références

[1] Loula Fezoui, Stéphane Lanteri, Stéphanie Lohrengel, and Serge Piperno, *Convergence and stability of a discontinuous galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes.* ESAIM: Mathematical Modelling and Numerical Analysis, 39(6):1149 1176, 2005.

[2] Jan S Hesthaven and Tim Warburton, *Nodal discontinuous Galerkin methods: algorithms, analysis, and applications.* Springer Science and Business Media, 2007.

[3] Robert I McLachlan and G Reinout W Quispel, *Splitting methods.* Acta Numerica, 11:341434, 2002.

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