Error estimates on ergodic properties of Feynman–Kac semigroups

Grégoire Ferré, CERMICS, Ecole Nationale des Ponts et Chaussées

Gabriel Stoltz, CERMICS, Ecole Nationale des Ponts et Chaussées

Feynman–Kac semigroups corresponds to stochastic processes whose trajectories are weighted depending on their path. They appear in several areas of mathematics, such as quantum physics (as Diffusion Monte Carlo), large deviations, or nonlinear filtering. Given that, in order to conserve probability, the expectation of such a process needs to be normalized, the PDE associated to its probability density is nonlinear. However, the limiting distribution ν_W is solution to the following linear eigenvalue problem

$$(\mathcal{L} + W)^\dagger \nu_W = \lambda \nu_W,$$

where $\mathcal{L} = \Delta + b \cdot \nabla$ is the generator of the stochastic process and W is a weight function. The main difference with the invariant measure of a stochastic process is the source term W, hence the eigenvalue associated with the invariant measure is not zero.
Typical finite elements methods generally fail to solve such an eigenproblem for large systems. This is a reason why Feynman–Kac semigroups are useful in practice: averages with respect to ν_W are estimated through ergodic averages over trajectories. However, for computing such a stochastic dynamics, one needs to discretize it in time. The discretized process then has an invariant measure $\nu_{W,\Delta t}$ and an eigenvalue $\lambda_{\Delta t}$ that differ from the correct ones, and it is interesting to quantify precisely this error depending on the numerical scheme at hand and the time step.
The purpose of [1] is to adress this issue by analytical techniques. With this approach based on semigroup expansions, we obtain that there exists $p \in \mathbb{N}^*$ such that for any test function φ:

$$\int \varphi \, d\nu_{W,\Delta t} = \int \varphi \, d\nu_W + \Delta t^p \int f \varphi \, d\nu_W + \mathrm{O}(\Delta t^{p+1}),$$

where f is solution to a *linear* Poisson equation. In words, we quantify the order of convergence and characterize the leading order correction. Compared to the usual case $W = 0$, one has to consider non probability-conserving generators, which raises a number of difficulties for which we develop specific tools. This particularity of the model is due to the fact that in general, $\lambda \neq 0$. This also led us to provide new estimators of λ with precise rates of convergence.
This work not only justifies results already observed numerically in the litterature, but also provides criteria to construct new efficient integration schemes of the Feynman–Kac dynamics. We believe that the approach developped in [1] can be used for other processes that do not conserve probability, like quasi-stationary distributions.

Références

[1] G. Ferré & G. Stoltz, *Error estimates on ergodic properties of Feynman–Kac semigroups*, ArXiv:1712.04013, 2017.

Grégoire Ferré, 6-8 Avenue Blaise Pascal, Champs sur Marne 77455
gregoire.ferre@enpc.fr