Enriched finite elements for high-frequency vibrations of geometrically heterogeneous bars and Timoshenko beams

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Introduction This study proposes a new enriched finite element method (FEM) to handle high-frequency vibrations of bars (traction-compression) and beams (bending) with varying cross-section. Indeed, analytical solutions are available for a limited number of geometries, especially for Timoshenko beams [2], and traditional h- and hp-FEM may become costly at increasing frequencies as they require a minimal *resolution* (number of elements per wavelength) to reach the convergence regime. Hereinafter, the focus is on bars, but numerical illustrations will be provided for both bars and beams.

Problem and approximation space Consider a bar of length L made of an homogeneous material, such that the wavespeed c is constant, and submitted to time-harmonic excitations with circular frequency ω. In a non-dimensional setting, the amplitude $u(x)$ of the longitudinal displacement obeys the wave equation $(\mathcal{A}u')' + \mathcal{A}k^2u = f$, where $\mathcal{A}(x)$ is the profile of the cross-section, k is the non-dimensional counterpart of ω/c and f is the amplitude of time-harmonic distributed forces. Boundary conditions, possibly including boundary excitations, complete the problem.
Let $\Omega_h = \{x_0, \dots, x_N\}$ be a mesh of the domain $[0, L]$. Following the ideas of the partition of unity method [1], we define the following enriched FE space:

$$V_h^\Psi = \text{Span}\left\{\{\varphi_n^0\}_{n=0\dots N} \bigcup \left\{\{\varphi_n^- \psi_n^m\} \bigcup \{\varphi_n^+ \psi_n^m\}\right\}_{n=1\dots N,\, m=1,2}\right\},$$

where the φ_n^0 are the traditional "hat" functions (the basis of the space of continuous and piecewise affine functions), the φ_n^- (resp. φ_n^+) are "half-hat" functions, i.e. the restriction of φ_{n+1}^0 (resp. φ_n^0) to the n-th element, and $\Psi_n = \{\psi_n^1, \psi_n^2\}$ is the enrichment family associated with this element. Two of these families are considered.

Global and local enrichments First, we use the family $\Psi_n^k = \{\sin(kx), \cos(kx)\}$ for all elements, i.e. the basis of solutions to the wave equation for constant profiles. We prove that using the corresponding enriched space provides the same convergence order than using a fifth-order polynomial basis, and show that *static condensation* is easily implemented thanks to the choice of "half-hat" functions as partition of unity, resulting in a well-conditioned linear system. On numerical tests, we see that (i) the resolution needed to reach convergence is much lower than for the polynomial basis, and (ii) the error decreases with the frequency for a given resolution, making our proposal well adapted to high-frequency problems. Finally, we consider the following element-dependent family:

$$\Psi_n^{k,\delta} = \left\{e^{-\delta_n x}\sin(\widetilde{k}_n x), e^{-\delta_n x}\cos(\widetilde{k}_n x)\right\}, \quad \text{with} \quad \delta_n = (\mathcal{A}'/2\mathcal{A})(x_{n-1/2}) \quad \text{and} \quad \widetilde{k}_n = \sqrt{k^2 - \delta_n^2},$$

which corresponds to the basis of solutions for exponential profiles $\mathcal{A}_\delta(x) = e^{-2\delta_n x}$, the value of δ_n being determined using a Taylor expansion of the wave equation about the middle $x_{n-1/2}$ of the n-th element. This second family enables to keep all the properties described above, while reducing again the FEM error by a factor up to 2 (depending on the profile $\mathcal{A}$) compared to the first family.

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Références

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