A Varifold Approach to Surface Approximation

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On the one hand, discrete geometry is a central and very topical issue, mainly because of its applications in computer graphics and image processing. There has been a huge literature on the subject in the last decade, leading to impressive numerical results. On the other hand, varifolds have been introduced by F. Almgren in 1965 in [1] to study *minimal surfaces*. Though mainly used with smooth surfaces and rectifiable sets, they are very flexible tools : both regular surfaces and discrete approximations (triangulations, point clouds, digital shapes ...) can be provided with a varifold structure, allowing to study surfaces and their different discretizations in a consistent unified setting. We aim at connecting these successful tools from geometric measure theory (varifolds) to practical issues in discrete geometry (surface comparison, notion of discrete curvature, geometric motions etc.).

Let us define varifolds. A *d–varifold* in an open set $\Omega \subset \mathbb{R}^n$ is a positive Radon measure on $\Omega \times G_{d,n}$ where $G_{d,n} = \{d\text{–vector plane of } \mathbb{R}^n\}$ is the d–Grassmannian. A varifold couples spatial and directional information: if M is a d–sub-manifold or more generally a d–rectifiable set provided with a multiplicity $\theta : M \to \mathbb{R}_+^*$, there is a natural d–varifold structure $v(M,\theta) = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ on (M,θ). Such varifolds are called rectifiable varifolds. Let us now give an example of varifold structure on a point cloud. Given $\{x_i\}_{i=1\ldots N} \subset \mathbb{R}^n$, weighted by masses $\{m_i\}_{i=1\ldots N} \subset \mathbb{R}_+$ and directions $\{P_i\}_{i=1\ldots N} \subset G_{d,n}$, we can define the d–varifold: $V_N = \sum_{i=1}^N m_i \, \delta_{x_i} \otimes \delta_{P_i}$.

A major issue is that, although the concept of varifold is very general, very few studies have been carried out outside the class of rectifiable varifolds. For instance, if we try to compute the so-called *first variation*, denoted by δV, of a point cloud varifold V in order to obtain a numerical estimate of the mean curvature of the underlying surface, we end up with a distribution of order 1, loosely speaking, we are differentiating Dirac masses. And yet, when dealing with a varifold V associated with a smooth surface $\delta V = -H\sigma$, where H is the mean curvature vector and σ is the surface measure, and hence identifies with a L^1 function. We propose to explain how these two ingredients – a weak formulation relying only on the varifold structure, and its consistency with the smooth setting – are sufficient to propose a notion of discrete mean curvature (actually a family of discrete mean curvatures associated with a regularization scale ϵ) having nice convergence properties and numerically easy to compute.

Elaborating on this approach, it is possible to adapt the process in order to recover not only a discrete mean curvature but a whole discrete second fundamental form. For this purpose, we regularize a slightly modified version of the *weak second fundamental form* introduced in [3] and based on the tangential divergence theorem evaluated with test functions defined on $\Omega \times G_{d,n}$. In [3], the conditions for the existence of a weak second fundamental form are too restrictive to apply to point cloud varifolds for instance, even up to a regularization process. To circumvent this drawback, we have to weaken this notion of weak second fundamental, at the expense of losing the regularity result following from its p integrability established by Hutchinson.

Références

[1] J. Almgren, *Theory of Varifolds*, Mimeographed lecture notes, 1965.

[2] B. Buet, G.P. Leonardi, S. Masnou, *A Varifold Approach to Surface Approximation*, ARMA, 2017.

[3] J.E. Hutchinson, *Second Fundamental Form for Varifolds and the Existence of Surfaces Minimising Curvature*, Indiana Univ. Math. J., (35)1, 1986.

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