Étude d'une méthode de volumes finis pour la résolution des équations de Maxwell en 2D

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Références

Système de Maxwell (mode T.M)

$$
\begin{aligned}
\frac{\partial B}{\partial t} + \nabla \times \mathbf{E} &= 0 && \textit{dans } [0, T] \times \Omega \\
\frac{\partial \mathbf{E}}{\partial t} - c^2 \boldsymbol{\nabla} \times B &= -\frac{1}{\epsilon_0}\mathbf{J} && \textit{dans } [0, T] \times \Omega \\
\alpha \mathbf{E} \cdot \mathbf{t} - \beta B &= 0 && \textit{dans } [0, T] \times \partial\Omega \\
\mathbf{E}(0, .) &= \mathbf{E}^0 && \textit{dans } \Omega \\
B(0, .) &= B^0 && \textit{dans } \Omega \\
\textit{avec} \quad \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} && \textit{dans } [0, T] \times \Omega
\end{aligned}
$$

Schéma de Yee

1/ $$\int_{C_i'} \left(\frac{\partial B}{\partial t} + \nabla \times \mathbf{E} \quad = 0 \right)$$

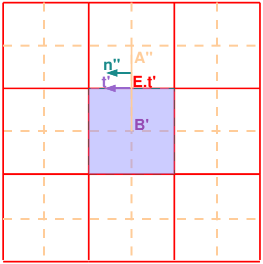
$$\Rightarrow \int_{C_i'} \frac{\partial B}{\partial t} + \sum_{j \in AC_i'} \int_{A_j'} \mathbf{E} \cdot \mathbf{t}_j' \quad = 0$$

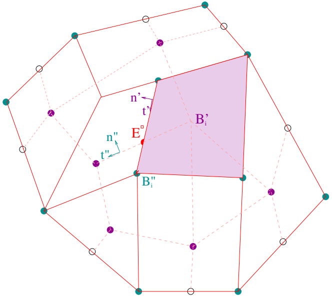
$$\rightsquigarrow |C_i'| \frac{\partial B_i'}{\partial t} + \sum_{j \in AC_i'} |A_j'| \mathbf{E}_j \cdot \mathbf{t}_j' \quad = 0$$

2/ $$\int_{A''_j} \left(\frac{\partial \mathbf{E}}{\partial t} - c^2 \boldsymbol{\nabla} \times B = -\frac{1}{\epsilon_0} \mathbf{J} \right) \cdot \mathbf{n}''_j$$

$$\Rightarrow \int_{A''_j} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{n}''_j - c^2 \int_{A''_j} \boldsymbol{\nabla} B \cdot \mathbf{t}''_j = -\frac{1}{\epsilon_0} \int_{A''_j} \mathbf{J} \cdot \mathbf{n}''_j$$

$$\rightsquigarrow |A''_j| \frac{\partial \mathbf{E}_j}{\partial t} \cdot \mathbf{t}_j' - c^2 (B_2' - B_1') = -\frac{1}{\epsilon_0} |A''_j| \mathbf{J}_j \cdot \mathbf{n}''_j$$



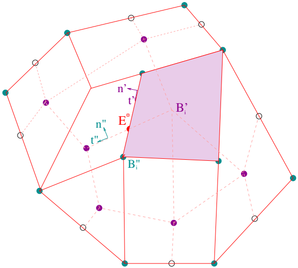


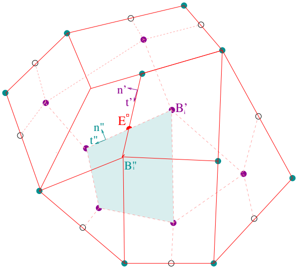
$$1/ \qquad |C'_i|\frac{\partial B'_i}{\partial t} + \sum_{j \in AC'_i} |A'_j| \mathbf{E}_j \cdot \mathbf{t}'_j \quad = 0$$

$$2/ \qquad |A"_j|\frac{\partial \mathbf{E}_j}{\partial t} \cdot \mathbf{n}"_j - c^2(B'_2 - B'_1) = -\frac{1}{\epsilon_0}|A"_j| \mathbf{J}_j \cdot \mathbf{n}"_j$$

problème : $\mathbf{n}"_j \neq \mathbf{t}'_j$!

Discretisation de la loi de Faraday sur le maillage primal et dual





$$\int_{C'_i}\left(\frac{\partial B}{\partial t}+\nabla\times\mathbf{E}^{\diamond}=0\right)$$

$$\hookrightarrow |C'_i|\frac{\partial B'_i}{\partial t}+\sum_{j\in AC'_i}|A'_j|\mathbf{E}^{\diamond}_j\cdot\mathbf{t}'_j=0$$

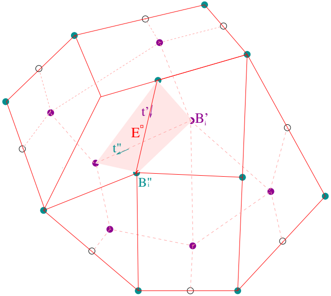
$$\hookrightarrow \frac{\partial B'_i}{\partial t}+(\nabla'\times\mathbf{E}^{\diamond})_i=0$$

$$\int_{C"_i}\left(\frac{\partial B}{\partial t}+\nabla\times\mathbf{E}^{\diamond}=0\right)$$

$$\hookrightarrow |C"_i|\frac{\partial B"_i}{\partial t}+\sum_{j\in AC"_i}|A"_j|\mathbf{E}^{\diamond}_j\cdot\mathbf{t}"_j=0$$

$$\hookrightarrow \frac{\partial B"_i}{\partial t}+(\nabla"\times\mathbf{E}^{\diamond})_i=0$$

Discretisation de la loi d'Ampere-Maxwell sur le maillage diamant



$$\int_{C_j^\diamond}\left(\frac{\partial \mathbf{E}}{\partial t} - c^2 \boldsymbol{\nabla}\times B = -\frac{1}{\epsilon_0}\mathbf{J}\right)$$

$$\rightsquigarrow |C_j^\diamond|\frac{\partial \mathbf{E}_j^\diamond}{\partial t} + c^2\left((B_2' - B_1')|A_j'|\mathbf{t}_j' + (B''_2 - B''_1)|A''_j|\mathbf{t}''_j\right) = -\frac{1}{\epsilon_0}|C_j^\diamond|\mathbf{J}_j^\diamond$$

$$\rightsquigarrow \frac{\partial \mathbf{E}_j^\diamond}{\partial t} - c^2\left(\boldsymbol{\nabla}^\diamond \times B'''^{\,n}\right)_j = -\frac{1}{\epsilon_0}\mathbf{J}_j^\diamond$$

$$B'^{\,n+1} = B'^{\,n} - \Delta t \nabla' \times \mathbf{E}^{\diamond n+1/2}$$

$$B''^{\,n+1} = B''^{\,n} - \Delta t \nabla'' \times \mathbf{E}^{\diamond n+1/2}$$

$$\mathbf{E}^{\diamond n+1/2} = \mathbf{E}^{\diamond n-1/2} + c^2 \Delta t \boldsymbol{\nabla}^{\diamond} \times B'''^{\,n} - \frac{\Delta t}{\epsilon_0} \mathbf{J}^{\diamond n}$$

Pour un maillage cartésien $\Downarrow$

$$B'^{\,n+1}_i = B'^{\,n}_i - \frac{\Delta t}{|C'_i|} \sum_{j \in \partial AC'_i} |A'_j| \mathbf{E}^{\diamond n+1/2}_j \cdot \mathbf{t}'_{ij}$$

$$B''^{\,n+1}_i = B''^{\,n}_i - \frac{\Delta t}{|C''_i|} \sum_{j \in AC''_i} |A''_j| \mathbf{E}^{\diamond n+1/2}{}_j \cdot \mathbf{t}''_{ij}$$

$$\mathbf{E}^{\diamond n+1/2}_j \cdot \mathbf{t}'_{ij} = \mathbf{E}^{\diamond n-1/2}_j \cdot \mathbf{t}'_{ij} - \frac{c^2 \Delta t}{|A''_j|} \left(B'_{i_j} - B'_i \right) - \frac{\Delta t}{\epsilon_0} \mathbf{J}^{\diamond n}_j \cdot \mathbf{t}'_{ij}$$

$$\mathbf{E}^{\diamond n+1/2}_j \cdot \mathbf{t}''_{ij} = \mathbf{E}^{\diamond n-1/2}_j \cdot \mathbf{t}''_{ij} - \frac{c^2 \Delta t}{|A'_j|} \left(B''_{i_j} - B''_i \right) - \frac{\Delta t}{\epsilon_0} \mathbf{J}^{\diamond n}_j \cdot \mathbf{t}''_{ij}$$

Discrétisation de la divergence sur le maillage primal et dual

$$\begin{aligned}
\frac{1}{|C'_i|}\int_{C'_i} \nabla\cdot\mathbf{E} &= \frac{1}{|C'_i|}\sum_{j\in AC'_i}\int_{A'_j}\mathbf{E}\cdot\mathbf{n}'_j \\
&\simeq \frac{1}{|C'_i|}\sum_{j\in AC'_i} |A'_j|\ \mathbf{E}^{\diamond}_j\cdot\ \mathbf{n}'_{ij} \\
&:= (\ \nabla'\cdot\ \mathbf{E}^{\diamond})_i \\
\frac{1}{|C"_i|}\int_{C"_i} \nabla\cdot\mathbf{E} &= \frac{1}{|C"_i|}\sum_{j\in AC"_i}\int_{A"_j}\mathbf{E}\cdot\mathbf{n}"_j \\
&\simeq \frac{1}{|C"_i|}\sum_{j\in AC"_i} |A"_j|\ \mathbf{E}^{\diamond}_j\cdot\ \mathbf{n}"_{ij} \\
&:= (\ \nabla"\cdot\ \mathbf{E}^{\diamond})_i
\end{aligned}$$

Propriétés des opérateurs discrets :

- $\nabla' \cdot (\boldsymbol{\nabla}^{\diamond} \times B'") = 0$
- $\nabla" \cdot (\boldsymbol{\nabla}^{\diamond} \times B'") = 0$
- $\langle \boldsymbol{\nabla}'" \times \mathbf{E}^{\diamond}, B'" \rangle_{'"} = \langle \mathbf{E}^{\diamond}, \boldsymbol{\nabla}^{\diamond} \times B'" \rangle_{\diamond} + \langle \mathbf{E}^{\diamond} \cdot \mathbf{t}', B'" \rangle_{\partial\Omega}$

où,

$$\langle H'", B'" \rangle_{'"} := \frac{1}{2} \left(\sum_{i=1}^{N'} |C'_i| \, H'_i \, B'_i + \sum_{i=1}^{N"} |C"_i| \, H"_i \, B"_i \right)$$

$$\langle \mathbf{E}^{\diamond}, \mathbf{H}^{\diamond} \rangle_{\diamond} := \sum_{i=1}^{N^{\diamond}} |C_i^{\diamond}| \, \mathbf{E}_i^{\diamond} \cdot \mathbf{H}^{\diamond}$$

$$\langle H^{\diamond}, B'" \rangle_{\partial\Omega} := \frac{1}{4} \sum_{A'_j \subset \partial\Omega} |A'_j| H_j^{\diamond} (B"_{j2} + B"_{j1} + 2B'_j)$$

.Discrétisation adéquate de ρ et $\mathbf{J}$
.Condition initiales satisfaisant la loi de Gauss discrète $\Longrightarrow$ Loi de Gauss discrète satisfaite à chaque instant

$$\nabla'.\mathbf{E}^{\diamond n+1/2} = \frac{\rho'^{n+1/2}}{\epsilon_0}$$

$$\nabla''.\mathbf{E}^{\diamond n+1/2} = \frac{\rho''^{n+1/2}}{\epsilon_0}$$

Discrétisation adéquate $\leftrightarrow$ Conservation de l'équation de charge discrète

$$\frac{\rho'^{n+1/2} - \rho'^{n-1/2}}{\Delta t} + \nabla'.\mathbf{J}^{\diamond n} = 0$$

$$\frac{\rho''^{n+1/2} - \rho''^{n-1/2}}{\Delta t} + \nabla''.\mathbf{J}^{\diamond n} = 0$$

Exemple d'une discrétisation adéquate de ρ et $\mathbf{J}$:

$$\rho_i'^{n+1/2} = \frac{1}{|C_i'|}\int_{C_i'} \rho(\mathbf{X}, t^{n+\frac{1}{2}})d\mathbf{X}$$

$$\rho''^{\,n+1/2}_{i} = \frac{1}{|C''_i|}\int_{C''_i} \rho(\mathbf{X}, t^{n+\frac{1}{2}})d\mathbf{X}$$

$$\mathbf{J}_i^{\diamond n} \cdot \mathbf{n}_i' = \frac{1}{\Delta t}\int_{t^{n-1/2}}^{t^{n+1/2}} \frac{1}{|A_i'|}\int_{A_i'} \mathbf{J}\cdot \mathbf{n}_i' d\mathbf{X}dt$$

$$\mathbf{J}_i^{\diamond n} \cdot \mathbf{n}''_i = \frac{1}{\Delta t}\int_{t^{n-1/2}}^{t^{n+1/2}} \frac{1}{|A''_i|}\int_{A''_i} \mathbf{J}\cdot \mathbf{n}''_i d\mathbf{X}dt$$

Conservation d'une énergie électromagnétique discrète :

$$\begin{aligned} \mathbb{E}^n &:= \frac{\epsilon}{2}\left(\left|\mathbf{E}^{\diamond n+1/2}\right|_{\diamond}^2 + c^2 \left\langle B^{\prime\prime\prime\, n}, B^{\prime\prime\prime\, n+1}\right\rangle_{\prime\prime\prime}\right) \\ &= \mathbb{E}^0 \end{aligned}$$

Stabilité :

- Schéma stable sous une condition CFL

$$c\Delta t < \min_i \min_{A'_j \subset \partial C'_i \setminus \partial\Omega} \sqrt{\frac{2|A''_j||C'_i|\sin\theta_j}{(1+|\cos\theta_j|)|\partial C'_i|}}$$

- condition CFL pour un maillage cartésien

$$\frac{c\Delta t}{h} \leq \frac{1}{\sqrt{2}}$$

$\Rightarrow$ CFL du schéma de Yee

Pour des champs réguliers

$$||er^n|| := \sqrt{\left|er^{\diamond n+1/2}\right|_{\diamond}^{2} + \frac{1}{2}\left(|er'^n|_{'}^{2} + |er''^{\,n}|_{''}^{2}\right)} \leq K(h + \Delta t^2)$$

Où,

$$\begin{aligned} er^n &:= \left(er'^n, er''^{\,n}, er^{\diamond n+1/2}\right) \\ er^{\diamond n+1/2} &:= \mathbf{E}^{\diamond n+1/2} - \Pi^{\diamond n+1/2}\mathbf{E} \\ er'^n &:= B'^n - \Pi'^n B \\ er''^{\,n} &:= B''^n - \Pi''^n B \end{aligned}$$

$\Pi^{\diamond n+1/2}\mathbf{E}$:Projection du champ électrique exact sur le maillage diamant
$\Pi'^n B$:Projection du champ magnétique exact sur le maillage primal
$\Pi''^n B$:Projection du champ magnétique exact sur le maillage dual

$$\frac{\Pi'^{n+1}B - \Pi'^{n}B}{\Delta t} + \nabla' \times \Pi^{\diamond n+1/2}\mathbf{E} = r'^{n}$$
$$\frac{\Pi''^{n+1}B - \Pi''^{n}B}{\Delta t} + \nabla'' \times \Pi^{\diamond n+1/2}\mathbf{E} = r''^{n}$$
$$\frac{\Pi^{\diamond n+1/2}\mathbf{E} - \Pi^{\diamond n-1/2}\mathbf{E}}{\Delta t} - c^2 \nabla^{\diamond} \times \Pi'''^{n}B + \frac{1}{\epsilon_0}\mathbf{J}^{\diamond n} = r^{\diamond n}$$

$$\Downarrow$$

$$\frac{er'^{n+1} - er'^{n}}{\Delta t} + \nabla' \times er^{\diamond n+1/2} = r'^{n}$$
$$\frac{er''^{n+1} - er''^{n}}{\Delta t} + \nabla'' \times er^{\diamond n+1/2} = r''^{n}$$
$$\frac{er^{\diamond n+1/2} - er^{\diamond n-1/2}}{\Delta t} - c^2 \nabla^{\diamond} \times er'''^{n} = r^{\diamond n}$$

$$
\begin{aligned}
er'^{\,n+1} &= er'^{\,n} - \nabla' \times er^{\diamond n+1/2} + \Delta t\, r'^{\,n} \\
er''^{\,n+1} &= er''^{\,n} - \nabla'' \times er^{\diamond n+1/2} + \Delta t\, r''^{\,n} \\
er^{\diamond n+1/2} &= er^{\diamond n-1/2} + c^2 \boldsymbol{\nabla}^{\diamond} \times er'''^{\,n} + \Delta t\, r^{\diamond n}
\end{aligned}
$$

$$\Downarrow$$

$$er^{n+1} = \mathcal{M}_h er^n + \Delta t\ r^n = \mathcal{M}_h^{n+1} er^0 + \Delta t \sum_{k=0}^{n} \mathcal{M}_h^{n-k} r^k$$

$$\Downarrow$$

$$||er^n|| \leq ||\mathcal{M}_h^{n+1} er^0|| + \Delta t \sum_{k=0}^{n} ||\mathcal{M}_h^{n-k} r^k|| \leq K \left(||er^0|| + \Delta t \sum_{k=0}^{n} ||r^k|| \right)$$

Projection vérifiant l'équation de Faraday discrète

$$\frac{1}{|C_i'|}\int_{C_i'}\int_{t^n}^{t^{n+1}}\left(\frac{\partial B}{\partial t}+\nabla\times\mathbf{E}=0\right)$$

$$\Downarrow$$

$$\frac{\Pi_1'^{n+1}B-\Pi_1'^{n}B}{\Delta t}+\nabla'\times\Pi_1^{\diamond n+1/2}\mathbf{E}=0$$

avec

$$(\Pi_1^{\diamond n+1/2}\mathbf{E})_j\cdot\mathbf{t}_j' := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\frac{1}{|A_j'|}\int_{A_j'}\mathbf{E}\cdot\mathbf{t}_j'$$

$$(\Pi_1'^{n}B)_i := \frac{1}{|C_i'|}\int_{C_i'}B(.,t^n)$$

$$\frac{1}{|C"_i|}\int_{C"_i}\int_{t^n}^{t^{n+1}}\left(\frac{\partial B}{\partial t}+\nabla\times\mathbf{E}=0\right)$$

$$\Downarrow$$

$$\frac{\Pi_1^{"n+1}B-\Pi_1^{"n}B}{\Delta t}+\nabla"\times\Pi_1^{\diamond n+1/2}\mathbf{E}=0$$

avec

$$(\Pi_1^{\diamond n+1/2}\mathbf{E})_j\cdot\mathbf{t}"_j \quad := \quad \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\frac{1}{|A"_j|}\int_{A"_j}\mathbf{E}\cdot\mathbf{t}"_j$$

$$(\Pi_1^{"n}B)_i \quad := \quad \frac{1}{|C"_i|}\int_{C"_i}B(.,t^n)$$

Projection vérifiant l'équation d'Ampère discrète

$$\frac{1}{|A'_j|}\int_{A'_j}\int_{t^{n-1/2}}^{t^{n+1/2}}\left(\frac{\partial \mathbf{E}}{\partial t} - c^2\nabla\times B = -\frac{1}{\epsilon_0}\mathbf{J}\right)$$

$$\Downarrow$$

$$\frac{\Pi_2^{\diamond n+1/2}\mathbf{E} - \Pi_2^{\diamond n-1/2}\mathbf{E}}{\Delta t} \quad - \quad c^2\nabla^{\diamond}\times\Pi'{''}{}_2^{n}B + \frac{1}{\epsilon_0}\mathbf{J}^{\diamond n} = 0$$

avec

$$(\Pi_2^{\diamond n+1/2}\mathbf{E})_j\cdot\mathbf{n}'_j \quad := \quad \frac{1}{|A'_j|}\int_{A'_j}\mathbf{E}(.,t^{n+1/2})\cdot\mathbf{n}'_j$$

$$(\Pi_2^{\diamond n+1/2}\mathbf{E})_j\cdot\mathbf{n}''_j \quad := \quad \frac{1}{|A''_j|}\int_{A''_j}\mathbf{E}(.,t^{n+1/2})\cdot\mathbf{n}''_j$$

$$(\Pi_2'^{n}B)_i := \frac{1}{\Delta t}\int_{t^{n-1/2}}^{t^{n+1/2}} B(C'_i,.) \quad et \quad (\Pi_2''^{n}B)_i := \frac{1}{\Delta t}\int_{t^{n-1/2}}^{t^{n+1/2}} B(C''_i,.)$$

$$\Pi^{\diamond n+1/2}\mathbf{E} := \Pi_1^{\diamond n+1/2}\mathbf{E}$$
$$\Pi'^{n}B := \Pi_2'^{n}B$$
$$\Pi''^{n}B := \Pi_2''^{n}B$$

$\Rightarrow$ Erreur initiale

$$||er^0|| \leq K(h+\Delta t^2)$$

$\Downarrow$

Erreur de troncature

$$r'^{n} = \frac{\Pi_2'^{n+1}B - \Pi_2'^{n}B - \Pi_1'^{n+1}B + \Pi_1'^{n}B}{\Delta t}$$

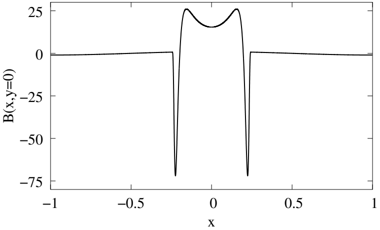
$$r''^{n} = = \frac{\Pi''^{n+1}_2 B - \Pi''^{n}_2 B - \Pi''^{n+1}_1 B + \Pi''^{n}_1 B}{\Delta t}$$

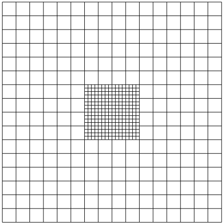
$$r^{\diamond n} = \frac{\Pi_1^{\diamond n+1/2}\mathbf{E} - \Pi_1^{\diamond n-1/2}\mathbf{E} - \Pi_2^{\diamond n+1/2}\mathbf{E} + \Pi_2^{\diamond n-1/2}\mathbf{E}}{\Delta t}$$

$\Downarrow$

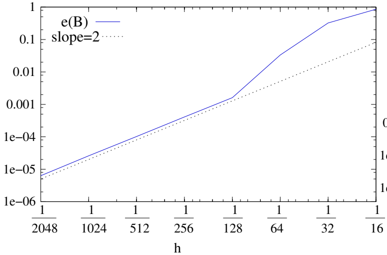
$$||r^n|| \leq K(h+\Delta t^2)$$

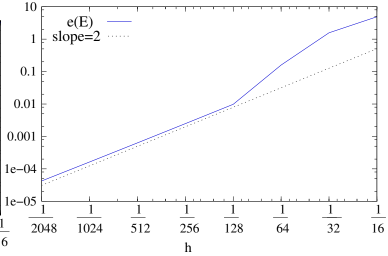
Etude de convergence sur des maillages non conformes





Convergence





$$e(B) = \sqrt{\frac{\int_{9T}^{10T} |B - \Pi B|_{L^2}^2}{\int_{9T}^{10T} |\Pi B|_{L^2}^2}},$$

B : Champs magnétique numérique

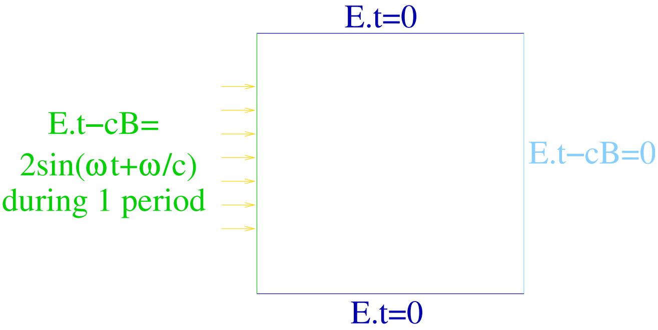
ΠB : Projection du champ magnétique

$$e(\mathbf{E}) = \sqrt{\frac{\int_{9T}^{10T} |\mathbf{E} - \Pi \mathbf{E}|_{L^2}^2}{\int_{9T}^{10T} |\Pi \mathbf{E}|_{L^2}^2}},$$

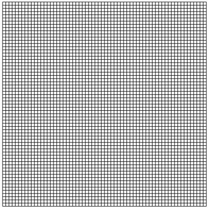
$\mathbf{E}$: Champs électrique numérique

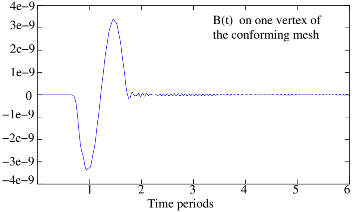
$\Pi\mathbf{E}$: Projection du champ électrique exact

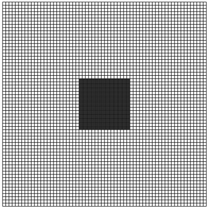
Onde rentrante avec sortie absorbante

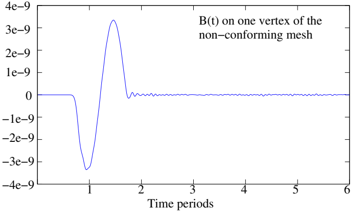


Reflexions parasites









$\Rightarrow$ La non conformité du maillage n'amplifie pas les reflexions parasites.

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