Convergence of approximation schemes for nonlocal front propagation equations

Aurélien Monteillet

Univ. Brest

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Outline

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Geometric equations

We are interested in geometric equations governing the movement of a family $K = \{K(t)\}_{t\in[0,T]}$ of compact subsets of $\mathbb{R}^N$:

$$V_{x,t} = h[K](x, t, \nu_{x,t}, A_{x,t}). \tag{1}$$

- $V_{x,t}$ is the normal velocity of a point x of $\partial K(t)$.
- $\nu_{x,t}$ is the unit exterior normal to $K(t)$ at $x \in \partial K(t)$.
- $A_{x,t} = [-\frac{\partial \nu_i}{\partial x_j}(x, t)]$ is the curvature matrix of $K(t)$ at $x \in \partial K(t)$.
- $K \mapsto h[K](x, t, \nu_{x,t}, A_{x,t})$ is a **non-local dependence** in the whole front K (up to time t).

Initial condition

Let $u_0 : \mathbb{R}^N \to \mathbb{R}$ be a bounded and Lipschitz continuous function on $\mathbb{R}^N$ which represents the initial front, *i.e.* such that

$$\{u_0 \geq 0\} = K_0 \quad \text{and} \quad \{u_0 = 0\} = \partial K_0$$

for some fixed compact set $K_0 \subset \mathbb{R}^N$.

We also assume that there exists $R_0 > 0$ such that

$$u_0 = -1 \quad \text{in } \mathbb{R}^N \setminus \bar{B}(0, R_0).$$

Level-set equation

The **level-set** equation associated to (1) is

$$\begin{cases} u_t(x,t) = H[\mathbf{1}_{\{u\geq 0\}}](x,t,Du,D^2u)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2}$$

where

$$\begin{aligned} & H[\mathbf{1}_{\{u\geq 0\}}](x,t,Du,D^2u)|Du(x,t)| \\ =& h[\{u \geq 0\}]\left(x,t,-\frac{Du}{|Du|},\frac{1}{|Du|}\left(I-\frac{Du\,Du^T}{|Du|^2}\right)D^2u\right)|Du(x,t)|. \end{aligned}$$

Main issue

Main problem : h is not necessarily monotone in K :

$$K \subset K' \text{ does not imply}$$

$$h[K](x, t, \nu, A) \leq h[K'](x, t, \nu, A).$$

$\Rightarrow$ No inclusion principle :

$$K_0 \subset K_0' \text{ does not imply } K(t) \subset K'(t) \text{ for all } t \geq 0.$$

$\Rightarrow$ The classical viscosity techniques for building a solution fail.

Example : dislocation dynamics

Recently, the dislocation dynamics model,

$$V_{x,t} = c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) + c_1(x, t)$$

of associated level-set equation

$$u_t(x, t) = \left[c_0(\cdot, t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x, t)\right] |Du(x, t)|,$$

has drawn a lot of attention.

- $c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) = \int_{K(t)} c_0(x - y, t)\, dy$ is a **nonlocal driving force.**
- c_1 is a prescribed driving force.

Abscence of sign of $c_0 \Rightarrow$ Non-monotone problem.

Definition of weak solutions

Let $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ be a continuous function. We say that u is a weak solution of (2) if there exists $\chi \in L^\infty(\mathbb{R}^N \times [0, T]; [0, 1])$ such that :

1. u is a L^1 viscosity solution of

$$\begin{cases} u_t(x,t) = H[\chi](x,t,Du,D^2u)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

2. For almost all $t \in [0, T]$,

$$\mathbf{1}_{\{u(\cdot,t)>0\}} \leq \chi(\cdot,t) \leq \mathbf{1}_{\{u(\cdot,t)\geq 0\}}.$$

Moreover, we say that u is a classical solution of (2) if in addition, for almost all $t \in [0, T]$ and almost everywhere in $\mathbb{R}^N$,

$$\{u(\cdot,t) > 0\} = \{u(\cdot,t) \geq 0\}.$$

References

Similar definitions (with existence results) can be found in :

- Giga, Goto, Ishii (SIAM J. Math. Anal., 1992) : Existence of weak solutions for a Fitzhugh-Nagumo system.
- Soravia, Souganidis (SIAM J. Math. Anal., 1996) : Phase-field theory for a Fitzhugh-Nagumo system.
- Hilhorst, Logak, Schätzle (Interfaces Free Bound., 2000) : Phase-field approach for the evolution law : $V_{x,t} = H_{x,t} - Vol(K)$.
- Barles, Cardaliaguet, Ley, Monneau (to appear in SIAM J. Math. Anal.) : Existence and uniqueness for dislocation dynamics.
- Barles, Cardaliaguet, Ley, M. : General existence of weak solutions.

Outline

We are now concerned with numerical approximation of weak solutions to (2).

This work is inspired by the general convergence result of Barles and Souganidis ('91) on monotone, stable and consistent schemes.

Notation

We define :

- A time step $h = T/n$ for some $n \in \mathbb{N}^*$.
- Space steps $\Delta_i = \lambda_i h$ for $\lambda_i > 0$ fixed ($i = 1, \dots, N$).
- For $(i_1, \dots, i_N) \in \mathbb{Z}^N$, $x_{i_1,\dots,i_N} = (i_1\Delta_1, \dots, i_N\Delta_N)$,
- The space grid $\Pi_h = \bigcup_{(i_1,\dots,i_N)\in\mathbb{Z}^N} x_{i_1,\dots,i_N}$,

Approximation schemes

We consider approximation schemes of the following form : for any $k \in \mathbb{N}$ such that $(k+1)h \le T$, and for any $x \in \Pi_h$, we set

$$\begin{cases} u_h(x,(k+1)h) = u_h(x,kh) + h\, H_h[\mathbf{1}_{\{u_h \ge 0\}}](x,kh,u_h(\cdot,kh)), \\ u_h(x,0) = u_0(x). \end{cases} \tag{3}$$

We finally extend u_h to a piecewise constant function on $\mathbb{R}^N \times [0,T]$ by setting for any (x,t),

$$u_h(x,t) = u_h(x_h,[t/h]h),$$

where for $x \in \mathbb{R}^N$,

$$x_h := ([x_1/\Delta_1 + 1/2]\Delta_1, \dots, [x_N/\Delta_N + 1/2]\Delta_N) \in \Pi_h.$$

Nonlocal dependance

The dependance $H_h[\chi](x, kh, u)$ is in fact :

- On $\chi(x_{i_1,\dots,i_N}, lh)$ for $(i_1, \dots, i_N) \in \mathbb{Z}^N$ and $0 \le l \le k$,
- and $u(x_{i_1,\dots,i_N}, kh)$ for $(i_1, \dots, i_N) \in \mathbb{Z}^N$.

We keep in mind that $H[\chi](x, kh, u)$ depends on the entire history $\chi(\cdot, lh)$ for l up to k.

Proof of convergence

The proof is based on the proof of the convergence result of Barles and Souganidis :

consider (h_n) such that $\mathbf{1}_{\{u_{h_n} \geq 0\}} \rightharpoonup \chi$ in L^∞ weak-$\star$,

and prove that

$$\underline{u} = \liminf_* \, u_{h_n} \quad \text{and} \quad \overline{u} = \limsup{}^* \, u_{h_n}$$

are repectively a L^1 viscosity supersolution and subsolution of

$$u_t = H[\chi](x, t, Du, D^2 u)|Du|.$$

Proof of convergence

The proof of this fact is inspired by a new stability result of Barles ('06), and taking into account weak convergence in time of the Hamiltonians : the condition is

$$\int_0^t H[\chi_n](x,s,p,A)ds \underset{n\to+\infty}{\longrightarrow} \int_0^t H[\chi](x,s,p,A)ds.$$

There are also links with the study of stochastic pde's by Lions and Souganidis ('98).

Assumptions

Our assumptions are the following :

(H1) : *H_h is monotone* :

$$u \leq v \;\Rightarrow\; u(x) + H_h[\chi](x, kh, u) \leq v(x) + H_h[\chi](x, kh, v).$$

(H2) : *H_h is stable* : there exists $L > 0$ such that

$$\|u_h\|_\infty \leq L \quad \text{independantly of } \chi.$$

(H3) : *H_h is consistent with H* : if $\chi_h \in L^\infty(\mathbb{R}^N \times [0, T]; [0, 1])$ is such that $\chi_h \rightharpoonup \chi$ weakly-$*$ in $L^\infty(\mathbb{R}^N \times [0, T]; [0, 1])$ as $h \to 0$, then

$$h \sum_{l=0}^{[t/h]-1} H_h[\chi_h](x_h, lh, \phi) \underset{h\to 0}{\longrightarrow} \int_0^t H[\chi](x, s, D\phi(x), D^2\phi(x))\, ds$$

locally uniformly for $t \in [0, T]$.

The main result is the following :

Theorem

*Let $(u_h)_h$ be defined by the scheme (3) satisfying assumptions **(H1)** to **(H3)**.*

Then there exists a sequence $h_n \to 0$ and $u \in C^0(\mathbb{R}^N \times [0,T];\mathbb{R})$ such that $u_{h_n} \to u$ locally uniformly in $\mathbb{R}^N \times [0,T]$, and u is a weak solution of (2).

If (2) has a unique weak solution u, then the whole sequence (u_h) converges locally uniformly to u.

Application : We obtain convergence of the scheme proposed by Alvarez, Carlini, Monneau and Rouy for dislocation dynamics.

Outline

The equation

This is a joint work with N. Forcadel.

We consider the example of dislocation dynamics with a mean curvature term :

$$V_{x,t} = H_{x,t} + c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) + c_1(x, t).$$

where $H_{x,t} = Tr(A_{x,t})$ is the mean curvature of $\partial K(t)$ at a point x.

Level-set equation

The corresponding level-set equation is

$$\begin{cases} u_t = \left[\mathrm{div}\left(\frac{Du}{|Du|}\right) + c_0(\cdot,t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)\right] |Du| \\ u(\cdot,0) = u_0 \end{cases} \tag{4}$$

Abscence of sign of $c_0 \Rightarrow$ Non-monotone problem.

Known results

Forcadel :

Short time existence and uniqueness of a viscosity solution, provided the initial shape is a graph or a Lipschitz curve.

Main issues

1. Can we provide weak solutions to (4) ?
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2. Does the mean curvature term have a regularizing effect ?

   We search for $\chi(\cdot, t)$ in the particular form $\mathbf{1}_{E(t)}$ with some regularity in time.

Minimizing movements

We build E as a minimizing movement for our evolution law : following Almgren, Taylor and Wang (1993), we discretize the equation

$$V_{x,t} = H_{x,t} + c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) + c_1(x,t) \tag{5}$$

in time. Let h be a time step.

We are going to construct a sequence of sets $E_h(k)$, for $k \in \mathbb{N}$ such that $kh \leq T$, whose evolution with k is a discretization of (5).

Discretization

More precisely, we wish to construct a sequence of sets $E_h(k)$ such that for all $x \in \partial E_h(k+1)$,

$$\pm \frac{d_{E_h(k)}(x)}{h} = H_{x,(k+1)h} + c_0(\cdot,(k+1)h) \star \mathbf{1}_{E_h(k+1)}(x) + c_1(x,(k+1)h),$$

where we take the $+$ sign if $x \notin E_h(k)$, the $-$ sign otherwise.

This corresponds to an **implicit time discretization** of

$$V_{x,t} = H_{x,t} + c_0(\cdot,t) \star \mathbf{1}_{K(t)}(x) + c_1(x,t).$$

Corresponding gradient flow

We construct $E_h(k+1)$ by seeing the equation

$$\pm\frac{d_{E_h(k)}(x)}{h} = H_{x,(k+1)h} + c_0(\cdot,(k+1)h) \star \mathbf{1}_{E_h(k+1)}(x) + c_1(x,(k+1)h)$$

as the Euler equation corresponding to the minimization of the functionnal

$$E \mapsto \mathcal{F}(h, k+1, E,\ E_h(k))$$

$$= P(E) + \frac{1}{h}\int_{E\Delta E_h(k)} d_{\partial E_h(k)}(x)\,dx$$
$$- \int_E \left(\frac{1}{2}c_0(\cdot,(k+1)h) \star \mathbf{1}_E(x) + c_1(x,(k+1)h)\right)\,dx.$$

Definition (Minimizing movement)

Let $E_0 \in \mathcal{P}$. We say that $E : [0, T] \to \mathcal{P}$ is a minimizing movement associated to $\mathcal{F}$ with initial condition E_0 if there exist $h_n \to 0^+$ and sets $E_{h_n}(k) \in \mathcal{P}$ for all $k \in \mathbb{N}$ verifying $kh_n \leq T$, such that :

1. $E_{h_n}(0) = E_0$.
2. For any $n, k \in \mathbb{N}$ with $(k+1)h_n \leq T$,

   $E_{h_n}(k+1)$ minimizes the functional $E \to \mathcal{F}(h_n, k+1, E, E_{h_n}(k))$.
3. For any $t \in [0, T]$, $E_{h_n}([t/h_n]) \to E(t)$ in $L^1(\mathbb{R}^N)$ as $n \to +\infty$.

Results

Under adapted regularity assumptions on c_0 and c_1, we obtained :

Theorem (Forcadel, M.)

Let $E_0 \in \mathcal{P}$ with $\mathcal{L}^N(\partial E_0) = 0$. Then :

1. *There exist **Hölder continuous** minimizing movements associated to $\mathcal{F}$ with initial condition E_0.*

2. *The corresponding solution u of*

$$\begin{cases} u_t = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right) + c_0(\cdot, t) \star \mathbf{1}_{E(t)}(x) + c_1(x, t)\right] |Du| \\ u(\cdot, 0) = u_0 \end{cases}$$

   is a weak solution of (4).

The main ingredients of proof are :

- A lower density bound for $\mathcal{F}$-minimizers : $P(E, B_\rho(x)) \geq \beta\rho^{N-1}$.
- A Distance-Volume comparison to estimate $|E_h(k+1)\Delta E_h(k)|$.
- A regularity result for $\mathcal{F}$-minimizers, so that the Euler-Lagrange equation corresponding to our minimizing procedure is the discretized equation.