Convergence of approximation schemes for nonlocal front propagation equations

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Outline

1. Introduction

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3. Dislocation dynamics with a mean curvature term
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3. Dislocation dynamics with a mean curvature term
Geometric equations

We are interested in geometric equations governing the movement of a family $K = \{K(t)\}_{t \in [0, T]}$ of compact subsets of $\mathbb{R}^N$:

$$V_{x,t} = h[K](x, t, \nu_{x,t}, A_{x,t}).$$ \hspace{1cm} (1)

- $V_{x,t}$ is the normal velocity of a point $x$ of $\partial K(t)$.
- $\nu_{x,t}$ is the unit exterior normal to $K(t)$ at $x \in \partial K(t)$.
- $A_{x,t} = [-\frac{\partial \nu_i}{\partial x_j}(x, t)]$ is the curvature matrix of $K(t)$ at $x \in \partial K(t)$.
- $K \mapsto h[K](x, t, \nu_{x,t}, A_{x,t})$ is a non-local dependence in the whole front $K$ (up to time $t$).
Let $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ be a bounded and Lipschitz continuous function on $\mathbb{R}^N$ which represents the initial front, i.e. such that

$$\{u_0 \geq 0\} = K_0 \quad \text{and} \quad \{u_0 = 0\} = \partial K_0$$

for some fixed compact set $K_0 \subset \mathbb{R}^N$.

We also assume that there exists $R_0 > 0$ such that

$$u_0 = -1 \quad \text{in} \ \mathbb{R}^N \setminus \bar{B}(0, R_0).$$
Level-set equation

The **level-set** equation associated to (1) is

\[
\begin{aligned}
 & u_t(x, t) = H[1_{\{u \geq 0\}}](x, t, Du, D^2u)|Du(x, t)| & \text{in } \mathbb{R}^N \times (0, T), \\
 & u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N,
\end{aligned}
\]

(2)

where

\[
\begin{aligned}
 H[1_{\{u \geq 0\}}](x, t, Du, D^2u)|Du(x, t)| \\
 = h[\{u \geq 0\}]
 \left( x, t, -\frac{Du}{|Du|}, \frac{1}{|Du|} \left( I - \frac{Du Du^T}{|Du|^2} \right) D^2u \right) |Du(x, t)|.
\end{aligned}
\]
Main problem: $h$ is not necessarily monotone in $K$:

$$K \subset K'$$

does not imply

$$h[K](x, t, \nu, A) \leq h[K'](x, t, \nu, A).$$

⇒ No inclusion principle:

$$K_0 \subset K'_0$$

does not imply $K(t) \subset K'(t)$ for all $t \geq 0$.

⇒ The classical viscosity techniques for building a solution fail.
**Example : dislocation dynamics**

Recently, the dislocation dynamics model,

\[ V_{x,t} = c_0(\cdot, t) \ast 1_{K(t)}(x) + c_1(x, t) \]

of associated level-set equation

\[ u_t(x, t) = \left[ c_0(\cdot, t) \ast 1_{\{u(\cdot, t) \geq 0\}}(x) + c_1(x, t) \right] |Du(x, t)|, \]

has drawn a lot of attention.

- \( c_0(\cdot, t) \ast 1_{K(t)}(x) = \int_{K(t)} c_0(x - y, t) \, dy \) is a **nonlocal driving force**.
  
- \( c_1 \) is a prescribed driving force.

Absence of sign of \( c_0 \Rightarrow \) Non-monotone problem.
**Definition of weak solutions**

Let \( u : \mathbb{R}^N \times [0, T] \to \mathbb{R} \) be a continuous function. We say that \( u \) is a weak solution of (2) if there exists \( \chi \in L^\infty(\mathbb{R}^N \times [0, T]; [0, 1]) \) such that:

1. \( u \) is a \( L^1 \) viscosity solution of

\[
\begin{cases}
    u_t(x, t) = H[\chi](x, t, Du, D^2u)|Du(x, t)| & \text{in } \mathbb{R}^N \times (0, T), \\
    u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N.
\end{cases}
\]

2. For almost all \( t \in [0, T] \),

\[
1_{\{u(\cdot, t) > 0\}} \leq \chi(\cdot, t) \leq 1_{\{u(\cdot, t) \geq 0\}}.
\]

Moreover, we say that \( u \) is a classical solution of (2) if in addition, for almost all \( t \in [0, T] \) and almost everywhere in \( \mathbb{R}^N \),

\[
\{u(\cdot, t) > 0\} = \{u(\cdot, t) \geq 0\}.
\]
Similar definitions (with existence results) can be found in:


- Hilhorst, Logak, Schätzle (Interfaces Free Bound., 2000) : Phase-field approach for the evolution law: \( V_{x,t} = H_{x,t} - \text{Vol}(K) \).


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We are now concerned with numerical approximation of weak solutions to (2).

This work is inspired by the general convergence result of Barles and Souganidis (’91) on monotone, stable and consistent schemes.
Convergence of approximation schemes for nonlocal equations

Notation

We define:

- A time step $h = T/n$ for some $n \in \mathbb{N}^*$.
- Space steps $\Delta_i = \lambda_i h$ for $\lambda_i > 0$ fixed ($i = 1, \ldots, N$).
- For $(i_1, \ldots, i_N) \in \mathbb{Z}^N$, $x_{i_1,\ldots,i_N} = (i_1 \Delta_1, \ldots, i_N \Delta_N)$.
- The space grid $\Pi_h = \bigcup_{(i_1,\ldots,i_N) \in \mathbb{Z}^N} x_{i_1,\ldots,i_N}$.
Approximation schemes

We consider approximation schemes of the following form: for any $k \in \mathbb{N}$ such that $(k + 1)h \leq T$, and for any $x \in \Pi_h$, we set

$$
\begin{cases}
    u_h(x, (k + 1)h) = u_h(x, kh) + h H_h[1_{\{u_h \geq 0\}}](x, kh, u_h(\cdot, kh)), \\
    u_h(x, 0) = u_0(x).
\end{cases}
$$

(3)

We finally extend $u_h$ to a piecewise constant function on $\mathbb{R}^N \times [0, T]$ by setting for any $(x, t)$,

$$
u_h(x, t) = u_h(x_h, [t/h]h),$$

where for $x \in \mathbb{R}^N$,

$$
x_h := ([x_1/\Delta_1 + 1/2] \Delta_1, \ldots, [x_N/\Delta_N + 1/2] \Delta_N) \in \Pi_h.
$$
Nonlocal dependance

The dependance $H_h[\chi](x, kh, u)$ is in fact:

- On $\chi(x_{i_1}, ..., i_N, lh)$ for $(i_1, \ldots, i_N) \in \mathbb{Z}^N$ and $0 \leq l \leq k$,

- and $u(x_{i_1}, ..., i_N, kh)$ for $(i_1, \ldots, i_N) \in \mathbb{Z}^N$.

We keep in mind that $H[\chi](x, kh, u)$ depends on the entire history $\chi(\cdot, lh)$ for $l$ up to $k$. 
Proof of convergence

The proof is based on the proof of the convergence result of Barles and Souganidis:

consider \((h_n)\) such that \(1_{\{u_{h_n} \geq 0\}} \rightharpoonup \chi\) in \(L^\infty\) weak-\(*\), and prove that

\[
\underline{u} = \lim \inf_* u_{h_n} \quad \text{and} \quad \overline{u} = \lim \sup_* u_{h_n}
\]

are respectively a \(L^1\) viscosity supersolution and subsolution of

\[
u_t = H[\chi](x, t, Du, D^2 u)|Du|.
\]
Proof of convergence

The proof of this fact is inspired by a new stability result of Barles ('06), and taking into account weak convergence in time of the Hamiltonians: the condition is

$$\lim_{n \to +\infty} \int_0^t H[\chi_n](x, s, p, A)ds = \int_0^t H[\chi](x, s, p, A)ds.$$

There are also links with the study of stochastic pde’s by Lions and Souganidis ('98).
Assumptions

Our assumptions are the following:

(H1): \( H_h \) is monotone:

\[ u \leq v \Rightarrow u(x) + H_h[\chi](x, kh, u) \leq v(x) + H_h[\chi](x, kh, v). \]

(H2): \( H_h \) is stable: there exists \( L > 0 \) such that

\[ \| u_h \|_\infty \leq L \] independantly of \( \chi \).

(H3): \( H_h \) is consistent with \( H \): if \( \chi_h \in L^\infty(\mathbb{R}^N \times [0, T]; [0, 1]) \) is such that \( \chi_h \rightharpoonup \chi \) weakly-* in \( L^\infty(\mathbb{R}^N \times [0, T]; [0, 1]) \) as \( h \to 0 \), then

\[
\lim_{h \to 0} h \sum_{l=0}^{[t/h]-1} H_h[\chi_h](x_h, lh, \phi) \to \int_0^t H[\chi](x, s, D\phi(x), D^2\phi(x)) \, ds
\]

locally uniformly for \( t \in [0, T] \).
The main result is the following:

**Theorem**

Let $(u_h)_h$ be defined by the scheme (3) satisfying assumptions (H1) to (H3).

Then there exists a sequence $h_n \to 0$ and $u \in C^0(\mathbb{R}^N \times [0, T]; \mathbb{R})$ such that $u_{h_n} \to u$ locally uniformly in $\mathbb{R}^N \times [0, T]$, and $u$ is a weak solution of (2).

If (2) has a unique weak solution $u$, then the whole sequence $(u_h)$ converges locally uniformly to $u$.

**Application**: We obtain convergence of the scheme proposed by Alvarez, Carlini, Monneau and Rouy for dislocation dynamics.
The equation

This is a joint work with N. Forcadel.

We consider the example of dislocation dynamics with a mean curvature term:

\[ V_{x,t} = H_{x,t} + c_0(\cdot, t) \star 1_{K(t)}(x) + c_1(x, t). \]

where \( H_{x,t} = Tr(A_{x,t}) \) is the mean curvature of \( \partial K(t) \) at a point \( x \).
The corresponding level-set equation is

\[
\begin{cases}
    u_t = \text{div} \left( \frac{Du}{|Du|} \right) + c_0(\cdot, t) \ast 1_{\{u(\cdot,t) \geq 0\}}(x) + c_1(x, t) |Du| \\
    u(\cdot, 0) = u_0
\end{cases}
\] (4)

Abscence of sign of $c_0 \Rightarrow$ Non-monotone problem.
Known results

Forcadel:

**Short time existence and uniqueness** of a viscosity solution, provided the initial shape is a graph or a Lipschitz curve.
Main issues

1. Can we provide weak solutions to (4)?

2. Does the mean curvature term have a regularizing effect?
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   Yes, thanks to a general existence theorem.

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   Yes, thanks to a general existence theorem.

2. Does the mean curvature term have a regularizing effect ?
   We search for $\chi(\cdot, t)$ in the particular form $1_{E(t)}$ with some regularity in time.
We build $E$ as a minimizing movement for our evolution law: following Almgren, Taylor and Wang (1993), we discretize the equation

$$V_{x,t} = H_{x,t} + c_0(\cdot, t) \ast 1_{K(t)}(x) + c_1(x, t)$$

(5)

in time. Let $h$ be a time step.

We are going to construct a sequence of sets $E_h(k)$, for $k \in \mathbb{N}$ such that $kh \leq T$, whose evolution with $k$ is a discretization of (5).
More precisely, we wish to construct a sequence of sets $E_h(k)$ such that for all $x \in \partial E_h(k + 1)$,

$$\pm \frac{d E_h(k)}{h}(x) = H_{x,(k+1)h} + c_0(\cdot, (k + 1)h) \star 1_{E_h(k+1)}(x) + c_1(x, (k + 1)h),$$

where we take the $+$ sign if $x \not\in E_h(k)$, the $-$ sign otherwise.

This corresponds to an **implicit time discretization** of

$$V_{x,t} = H_{x,t} + c_0(\cdot, t) \star 1_{K(t)}(x) + c_1(x, t).$$
Corresponding gradient flow

We construct $E_h(k + 1)$ by seeing the equation

$$\pm \frac{dE_h(k)}{h}(x) = H_{x,(k+1)h} + c_0(\cdot, (k + 1)h) \ast 1_{E_h(k+1)}(x) + c_1(x, (k + 1)h)$$

as the Euler equation corresponding to the minimization of the functionnal

$$E \mapsto \mathcal{F}(h, k + 1, E, E_h(k))$$

$$= P(E) + \frac{1}{h} \int_{E \Delta E_h(k)} d\partial E_h(k)(x) \, dx$$

$$- \int_E \left( \frac{1}{2} c_0(\cdot, (k + 1)h) \ast 1_E(x) + c_1(x, (k + 1)h) \right) \, dx.$$
Definition (Minimizing movement)

Let $E_0 \in \mathcal{P}$. We say that $E : [0, T] \rightarrow \mathcal{P}$ is a minimizing movement associated to $\mathcal{F}$ with initial condition $E_0$ if there exist $h_n \rightarrow 0^+$ and sets $E_{h_n}(k) \in \mathcal{P}$ for all $k \in \mathbb{N}$ verifying $kh_n \leq T$, such that:

1. $E_{h_n}(0) = E_0$.

2. For any $n, k \in \mathbb{N}$ with $(k + 1)h_n \leq T$,

   $E_{h_n}(k + 1)$ minimizes the functional $E \rightarrow \mathcal{F}(h_n, k + 1, E, E_{h_n}(k))$.

3. For any $t \in [0, T], E_{h_n}([t/h_n]) \rightarrow E(t)$ in $L^1(\mathbb{R}^N)$ as $n \rightarrow +\infty$. 
Under adapted regularity assumptions on $c_0$ and $c_1$, we obtained:

**Theorem (Forcadel, M.)**

Let $E_0 \in \mathcal{P}$ with $\mathcal{L}^N(\partial E_0) = 0$. Then:

1. There exist **Hölder continuous** minimizing movements associated to $\mathcal{F}$ with initial condition $E_0$.

2. The corresponding solution $u$ of

   \[
   \begin{cases}
   u_t = \left[ \text{div} \left( \frac{Du}{|Du|} \right) + c_0(\cdot, t) \ast 1_{E(t)}(x) + c_1(x, t) \right] |Du| \\
   u(\cdot, 0) = u_0
   \end{cases}
   \]

   is a weak solution of (4).
The main ingredients of proof are:

- A lower density bound for $\mathcal{F}$-minimizers: $P(E, B_\rho(x)) \geq \beta \rho^{N-1}$.
- A Distance-Volume comparison to estimate $|E_h(k + 1) \Delta E_h(k)|$.
- A regularity result for $\mathcal{F}$-minimizers, so that the Euler-Lagrange equation corresponding to our minimizing procedure is the discretized equation.