Maximum principle for stationary solutions of a nonlocal nonlinear diffusion equation

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Abstract

We consider quasilinear elliptic equation where diffusion depends on nonlocal interactions governed by a parameter. We show in a radial symmetric setting, the existence of a maximum principle.

Résumé

Nous étudions une équation elliptique quasilinéaire dans laquelle la diffusion est paramétré par la longueur de différentes interactions non locales. Nous prouvons en considérant les solutions radiales symétriques l'existence d'un principe du maximum.

Introduction

Let Ω be a open bounded subset of $\mathbb{R}^n$, we define by L is diameter and $f \in H^{-1}(\Omega)$. For any x in Ω and r in $[0, +\infty)$, we define the nonlocal functional

$$l_r(.)(x) : L^2(\Omega) \to \mathbb{R}, \quad u \mapsto l_r(u)(x) = \int_{\Omega \cap B(x,r)} u(y)dy,$$

where $B(x, r)$ is the closed ball of $\mathbb{R}^n$ with radius r and centered at x. It is clear that $l_0 = 0$ et $l_r = l_L$ pour tout $r \geq L$, which allows us to restrict our study to the case where $r \in [0, L]$. The nonlocal functional $l_r(.)(x)$ can also be considered in a more general form by

$$l_r(.)(x) : L^2(\Omega) \to \mathbb{R}, \quad u \mapsto l_r(u)(x) = \int_{\Omega \cap B(x,r)} g(x,y)u(y)dy,$$

with $esssup \int_\Omega g^2(x, y)dy < \infty$ (see). In this work we would like to investigate stationary solutions of the nonlocal problem

$$u_t - div(a(l_r(u(t)))\nabla u) = f. \tag{1}$$

From a physical point of view, the problem(1) could describe a density of population of bacterias $u(x, t)$ subjected to a diffusion rate proportional to $a(l_r(u))$. In these kinds of issues $l_r(u)$ could be the total population of a subarea $B(x, r) \cap \Omega$ of Ω. For more details of modelisation see [1],[2] and [4] . By stationnary problem associated to (1) we mean the problem of finding $u = u(x)$ solution to

$$(P_r) \begin{cases} -div(a(l_r(u))\nabla u) = f \quad dans \quad H^{-1}(\Omega) \\ u \in H_0^1(\Omega) \end{cases} \tag{2}$$

where $a \in C(\mathbb{R}, \mathbb{R})$, $\inf_{\mathbb{R}} a > 0$, $\sup_{\mathbb{R}} a < \infty$. When it is necessary , we will take $a \in W^{1,\infty}(\mathbb{R})$.

We study first briefly the question of counting the stationary solutions when $r = L$.Then, we give a result of existence and uniqueness when r range from 0 to L. Finally we present a maximum principle result and numerical application.

The case $r \geq L$

- The following result is due to Michel Chipot and B. Lovat [2].

Assume that ϕ is the weak solution to

$$\begin{cases} -\Delta\phi = f \qquad dans \quad H^{-1}(\Omega) \\ \phi \in H_0^1(\Omega) \end{cases} \tag{3}$$

then the problem (2) has as many solutions as the problem in $\mathbb{R}$

$$a(\mu)\mu = l_L(\phi) \tag{4}$$

with $l_L(u) = \mu$

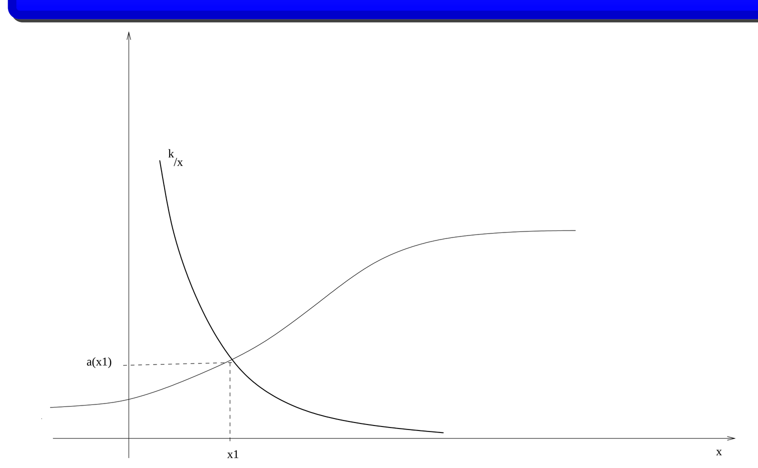

Figure 1: Uniqueness case

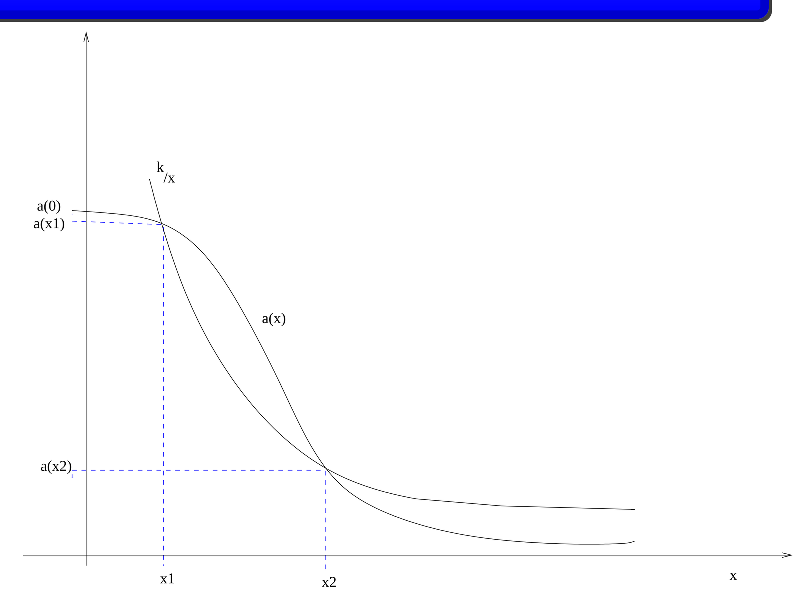

Figure 2: Multiple solutions

Existence and uniqueness result

If a is such that

$$|a(z_1) - a(z_2)| \leq \gamma|z_1 - z_2| \quad \forall (z_1, z_2) \in \mathbb{R}^2 \tag{5}$$

for all γ such that

$$\gamma < \frac{inf_{\mathbb{R}} a}{|f|_\star C(\Omega)} \tag{6}$$

where $C(\Omega)$ is a constant dependent to Ω, then problem (2) has a unique solution.

Maximum principle result

- Assume now that Ω is the open ball of $\mathbb{R}^n$ with radius $L/2$ centered at 0. We denote by $L_r^2(\Omega)$ the subspace of $L^2(\Omega)$ compound of radial solutions, we denote by $\tilde{u}(x) = u(\|x\|)$ where $\tilde{u}$ is a function of $L^2(0, L/2)$, see also [4] for more details .

Assume that $f \in L_r^2(\Omega)$, $f \geq 0$ a.e in Ω , in addition a is such that , there exists a solution μ_L of (4) such that $a(\mu_L) = sup_{[0,\infty)} a$ and $a(0) = inf_{[0,\infty)} a$ then any radial solution u_r of (P_r) satisfies

$$u_L \leq u_r \leq u_0. \tag{7}$$

Numerical Applications

- To illustrate our result in a simple case we took $n = 1$, $L = 2$, $f(x) = x^2$ et $a(x) = exp(x) + 0.1$. We represented the different values at the point $u_r(N/2)$ to put in a obvious place evolution of the solution u_r when r range 0 to L.

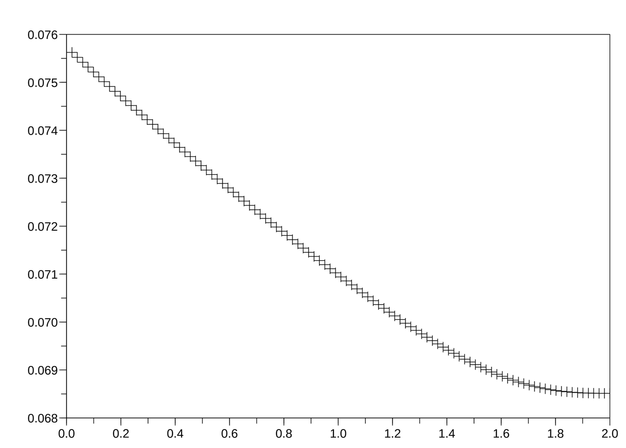

Figure 3: $u_r(N/2)$ according to r

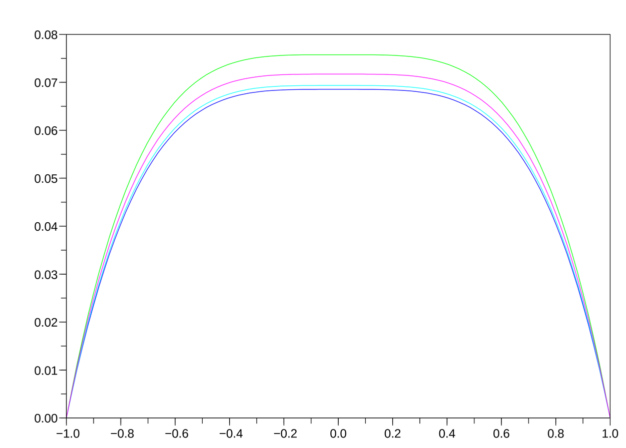

Figure 4: u_r according to r

Remarks

- Monotonicity case

It is clear that when a is nondecreasing, for all $r \geq L$ the problem (2) admits a unique solution. Under these assumptions , numerical simulations show us uniqueness for all $r \in [0, L]$, but theoretical proof remaining unknown.

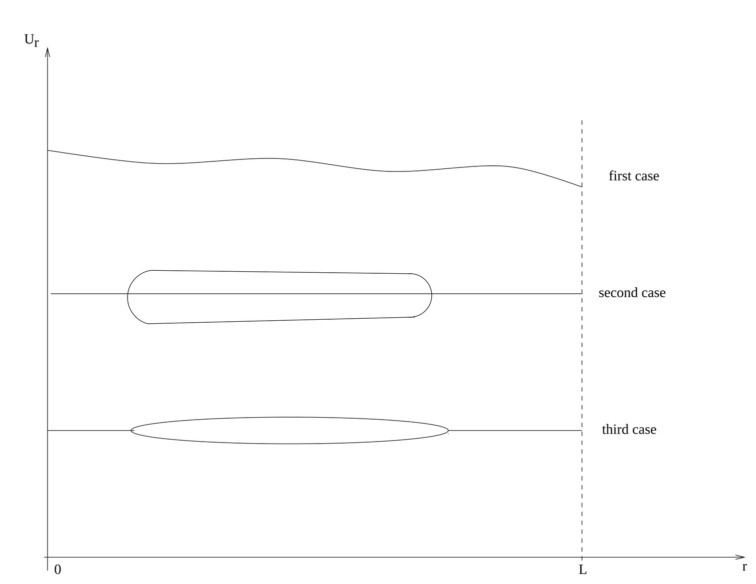

Figure 5: Possible bifurcation diagrams of u_r according to r

References

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[3] Emmanuel Chasseigne, Manuela Chaves and Julio D.Rossi, *Asymptotic behaviour for nonlocal diffusion equations*, J.Math. Pures Appl.(9),86(3):271-291,2006.

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