Viscosity Solutions of Fully Nonlinear Path Dependent PDEs

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Parabolic nonlinear path-dependent PDEs

- $\Omega = \{\omega \in C^0([0,T],\mathbb{R}^d), \omega_0 = 0\}$, $\|\omega\| = \sup_{t\le T}|\omega_t|$
- B canonical process, $\mathbb{F}$ the corresponding filtration

Our objective : wellposedness theory for the equation :

$$\big\{-\partial_t u - G(.,u,\partial_\omega u,\partial^2_{\omega\omega}u)\big\}(t,\omega) = 0 \quad \text{for} \quad t<T,\ \omega\in\Omega$$
$$u(T,\omega) = \xi(\omega)$$

where $\xi(\omega) = \xi\big((\omega_s)_{s\le T}\big)$ and $G(t,\omega,y,z,\gamma)$ is $\mathbb{F}-$prog. meas.

$$G : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d \longrightarrow \mathbb{R}, \quad \searrow \ \text{in} \ \gamma,$$

and the unknown process $u(t,\omega) = u(t,\omega_{t\wedge .})$ is prog. meas.

Previous literature

- Sobolev solutions :
  - Semilinear case : Pardoux and Peng (1990) and all subsequent literature on Backward SDEs
  - Fully nonlinear : Cheridito, Soner, T. and Victoir (2006), Soner, T. and Zhang (2010), second order backward SDEs

- Classical solutions for linear PPDE : Dupire 2009, Cont and Fournie (2010)

- Viscosity solutions :
  - Lukuyanov 2007 : First order path-dependent PDE
  - Ekren, T. and Zhang : this presentation

Time and space derivatives

- Time derivative introduced by Dupire :

$$\partial_t \varphi(t,\omega) := \lim_{h \searrow 0} \frac{\varphi(t+h, \omega_{t\wedge .}) - \varphi(t,\omega)}{h} \quad \text{if exists}$$

- Space derivatives : $\varphi(t,\omega) \in C^{1,2}$ if there exist continuous process, denoted $\partial_\omega \varphi$, $\partial^2_{\omega\omega}\varphi$, such that Itô's formula holds :

$$d\varphi \quad = \quad \partial_t \varphi dt + \frac{1}{2}\partial^2_{\omega\omega}\varphi : d\langle B\rangle + \partial_\omega \varphi \cdot dB \quad (\ldots)$$

Remark Dupire derivatives, Malliavin calculus

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Remark Dupire derivatives, Malliavin calculus

Example : Conditional expectation and Heat equation

- $\mathbb{P}_0$: Wiener measure on Ω
- By using the r.c.p.d. define for $\xi \in \mathbb{L}^1(\mathbb{P}_0)$:

$$u(t,\omega) := \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[\xi\right] \quad \text{for all} \quad t \le T,\ \omega \in \Omega$$

- Assume that $u \in C^{1,2}$, then :

$$du_t = \left(\partial_t u_t + \frac{1}{2}\partial^2_{\omega\omega} u_t\right)dt + \partial_\omega u_t dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

Since u is a $\mathbb{P}_0-$martingale, we obtain the heat equation :

$$\partial_t u + \frac{1}{2}\partial^2_{\omega\omega} u = 0 \quad \text{and} \quad u_T = \xi$$

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Example 2 : Stochastic control and HJB equation

- Control : $\{\kappa_r, r \leq T\}$, $\mathbb{F}-$prog. meas. valued in $K \subset \mathbb{R}^n$
- Controlled state : X^κ defined by :

$$X_r = x_r, r \leq t \quad \text{and} \quad dX^\kappa_s = \mu(s, X^\kappa{}_., \kappa_s)ds + \sigma(s, X^\kappa{}_., \kappa_s)dB_s$$

- Stochastic control problem :

$$u(t,\omega) := \sup_{\kappa \in \mathbb{K}} \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[\xi(X^\kappa{}_.)\right] \quad \text{for all} \quad t \leq T, \; \omega \in \Omega$$

- Assume that $u \in C^{1,2}$, then by dynamic programming :

$$-\partial_t u - \sup_{k \in K}\left\{\mu(t,\omega,k)\cdot\partial_\omega u + \frac{1}{2}\sigma\sigma^{\mathrm{T}}(t,\omega,k) : \partial^2_{\omega\omega} u\right\} = 0 \text{ and } u_T = \xi$$

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Other examples

- Stochastic PDEs : Lions and Souganidis use a transformation along the inverse of the characteristics equation $\longrightarrow$ forward path-dependent semilinear PDE

- Backward SDE $\longrightarrow$ path-dependent semilinear PDE

- Second order Backward SDE $\longrightarrow$ a certain class of path-dependent fully nonlinear PDE

- Dynamic programming equations for non-Markov differential games $\Longrightarrow$ Path-dependent Isaacs equation

Why viscosity solutions of PPDEs

• To obtain wellposedness for a larger class of equations

• Powerful stability result (in particular, analysis of numerical approximation)

• Easy, and very adapted to control and optimization problems

Main difficulty : the paths space Ω is not locally compact

Outline

1. Motivation and examples
   - Parabolic nonlinear PPDEs
   - Examples
2. Definitions and first properties
   - Smooth processes
   - Definition of viscosity solutions
   - Consistency, stability, partial comparison
3. Wellposedness
   - Additional assumption
   - Existence and uniqueness

Nondominated singular measures

- $L > 0$, $\mathcal{P}_L$: set of prob. meas. $\mathbb{P}$ on Ω s.t.

$$|\alpha^{\mathbb{P}}| \le L, \quad 0 \le \beta^{\mathbb{P}} \le \sqrt{2L}\, I_d, \quad dB_t = \beta_t^{\mathbb{P}} dW_t^{\mathbb{P}} + \alpha_t^{\mathbb{P}} dt, \quad \mathbb{P}\text{-a.s.}$$

for some $\mathbb{F}-$prog. meas. processes $\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}$, and some d-dimensional $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$

- For $\xi \in \mathbb{L}^1(\mathcal{F}_T, \mathcal{P}_L)$, define the nonlinear expectation :

$$\overline{\mathcal{E}}^L[\xi] = \sup_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}[\xi] \quad \text{and} \quad \underline{\mathcal{E}}^L[\xi] = \inf_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}[\xi] = -\overline{\mathcal{E}}^L[-\xi]$$

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Differentiability of processes

- For $\varphi \in C^0(\Lambda)$, the right time-derivative is defined by Dupire :

$$\partial_t \varphi(t,\omega) := \lim_{h\to 0, h>0} \frac{1}{h}\Big[\varphi(t+h, \omega_{\cdot\wedge t}) - \varphi(t,\omega)\Big], \quad t < T$$
$$\partial_t \varphi(T,\omega) := \lim_{t<T, t\uparrow T} \partial_t \varphi(t,\omega)$$

whenever the limits exist

- $\varphi \in C^{1,2}(\Lambda)$ if $\varphi \in C^0(\Lambda)$, $\partial_t \varphi \in C^0(\Lambda)$, and there exist $\partial_\omega \varphi \in C^0(\Lambda, \mathbb{R}^d)$, $\partial^2_{\omega\omega} \varphi \in C^0(\Lambda, \mathbb{S}^d)$ s.t. for all $\mathbb{P} \in \cup_{L>0} \mathcal{P}_L$:

$$d\varphi_t = \partial_t \varphi_t dt + \partial_\omega \varphi_t \cdot dB_t + \frac{1}{2} \partial^2_{\omega\omega} \varphi_t : d\left\langle B \right\rangle_t, \quad \mathbb{P}\text{-a.s.}$$

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Smooth test processes

Recall nonlinear Snell envelope :

$$\overline{\mathcal{S}}_t^L[X](\omega) := \sup_{\tau\in\mathcal{T}^t} \overline{\mathcal{E}}_t^L\left[X_\tau^{t,\omega}\right], \quad \text{and} \quad \underline{\mathcal{S}}_t^L[X](\omega) := -\overline{\mathcal{S}}_t^L[-X](\omega)$$

Define :

$$\underline{\mathcal{A}}^L u(t,\omega) := \Big\{\varphi \in C^{1,2}(\Lambda^t) : \quad (\varphi - u^{t,\omega})_t(\mathbf{0}) = \underline{\mathcal{S}}_t^L\left[(\varphi - u^{t,\omega})_{.\wedge \mathrm{h}}\right] \text{ for some } \mathrm{h} \in \mathcal{H}^t\Big\}$$

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Definition : Viscosity subsolution

$u \in \underline{\mathcal{U}}$ is a

- viscosity L-subsolution of PPDE if :

$$-\partial_t \varphi_t(0) - G\big(t, \omega, u(t,\omega), \partial_\omega \varphi_t(0), \partial^2_{\omega\omega}\varphi_t(0)\big) \leq 0$$

$$\text{for all } \; (t,\omega) \in [0,T) \times \Omega \text{ and } \varphi \in \underline{\mathcal{A}}^L u(t,\omega)$$

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Nonlinearity, first assumption

Assumption G1 $G(t, \omega, y, z, \gamma)$ nondecreasing in γ and satisfies :

(i) $G(\cdot, y, z, \gamma)$ is $\mathbb{F}$-prog. meas., and $\|G(\cdot, 0, 0, 0)\|_\infty < \infty$.

(ii) G is uniformly continuous in ω

(iii) G is uniformly Lipschitz in (y, z, γ)

Consistency with classical solutions

Theorem

Let Assumption G1 hold and $u \in C_b^{1,2}(\Lambda)$. Then the following assertions are equivalent :

- *u classical solution (resp. subsolution, supersolution) of PPDE*
- *u viscosity solution (resp. subsolution, supersolution) of PPDE*

Stability

Theorem

Let $(G^\varepsilon, \varepsilon > 0)$ *be a family of coefficients*

- *satisfying Assumptions G1 unif.*
- $G^\varepsilon \longrightarrow G$ *as* $\varepsilon \to 0$*, loc. unif.*

For fixed $L > 0$*, let* $(u^\varepsilon)_{\varepsilon > 0}$ *be such that*

- u^ε *is viscosity* $L-$*subsolution of PPDE with coefficients* G^ε*, for all* $\varepsilon > 0$,
- $u^\varepsilon \longrightarrow u$*, uniformly in* Λ.

Then u *is a viscosity* $L-$*subsolution of PPDE with coefficient* G.

Existence in all previous examples...

- Non-Markov stochatsic control
- Non-Markov stochatsic differential game
- Second order Backward SDE

all define a viscosity solution of the corresponding path-dependent PDE

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Freezing ω in the generator

- Define the deterministic function on $[t,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d$:

$$g^{t,\omega}(s,y,z,\gamma) \quad := \quad G(s,\omega_{\cdot\wedge t},y,z,\gamma)$$

- Consider the standard PDE :

$$\mathbf{L}^{t,\omega}v \quad := \quad -\partial_t v - g^{t,\omega}(s,v,Dv,D^2v) \;=\; 0, \quad (t,x)\in O_t^{\varepsilon,\eta}$$

where

$$O_t^{\varepsilon,\eta} \quad := \quad [t,(1+\eta)T)\times\{x\in\mathbb{R}^d : |x|<\varepsilon\}, \quad \varepsilon>0,\eta\geq 0$$

Additional Assumption on the generator

Assumption G2 For any small $\varepsilon > 0, \eta \geq 0$ and any $(t,\omega) \in \Lambda$, PDE is wellposed in the following sense :

(i) for all $\delta > 0$, there exist classical supersolution $\overline{w}^\delta$ and subsolution $\underline{w}^\delta$ of the frozen path PDE s.t.

$$-\|h\|_\infty - \delta \leq \underline{w}^\delta \leq h \leq \overline{w}^\delta \leq \|h\|_\infty + \delta \quad \text{on} \quad \partial O_t^{\varepsilon,\eta},$$

(ii) Peron's approach : given a continuous function $h : \partial O_t^{\varepsilon,\eta} \to \mathbb{R}$, the PDE with boundary condition h has a unique viscosity solution v and it satisfies $v = \overline{v} = \underline{v}$, where

$$\overline{v}(s,x) := \inf\Big\{\phi(s,x) : \phi \in C^{1,2}(\bar{O}_t^{\varepsilon,\eta}),\ \mathbf{L}^{t,\omega}\phi \geq 0 \text{ in } O_t^{\varepsilon,\eta},\ \phi \geq h \text{ on } \partial O_t^{\varepsilon,\eta}\Big\}$$

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Sufficient conditions for Assumption G2

Assumption G1 and

- G convex in γ and uniformly elliptic,
- or, G is convex in (y, z, γ)
- or, $d \leq 2$

The main results

Theorem (Comparison)

Under Assumptions G1-G2, let $u^1 \in \underline{\mathcal{U}}$, $u^2 \in \overline{\mathcal{U}}$, $\xi \in \mathrm{UC}_b(\Omega)$ *s.t*

- u^1 *is a bounded viscosity subsolution of PPDE*
- u^2 *is a bounded viscosity supersolution of PPDE*
- $u^1(T,\cdot) \leq \xi \leq u^2(T,\cdot)$

Then $u^1 \leq u^2$ *on* Λ.

Theorem (Existence)

Under Assumptions G1, G2, for any $\xi \in \mathrm{UC}_b(\Omega)$, *the PPDE with terminal condition* ξ *has a unique bounded viscosity solution* $u \in \mathrm{UC}_b(\Lambda)$.

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