Conic Optimization: An Exciting Present and a Promising Future

Miguel F. Anjos

Professor and Canada Research Chair

POLYTECHNIQUE MONTRÉAL

GERAD

Various parts are joint work with

A. Engau (U. Colorado-Denver)
B. Ghaddar (U. Waterloo; now IBM)
P. Hungerländer (Klagenfurt U.)
A. Vannelli (U. Guelph)
J.C. Vera (Tilburg U.)

EUROPT 2014 – Université de Perpignan, France – July 11, 2014

Conic Optimization

The problem of optimizing a linear or convex quadratic function over the intersection of an affine space and a pointed closed convex cone:

$$
\begin{array}{llll}
\text{(P)} & \inf & \langle c, x \rangle \\
 & \text{s.t.} & \langle a_i, x \rangle = b_i, i = 1, \ldots, m \\
 & & x \in \mathcal{K}
\end{array}
\qquad
\begin{array}{llll}
\text{(D)} & \sup & b^T y \\
 & \text{s.t.} & \sum_{i=1}^{m} y_i a_i + s = c \\
 & & s \in \mathcal{K}^*
\end{array}
$$

where the dual cone $\mathcal{K}^*$ is defined as

$$\mathcal{K}^* := \{\mathbf{y} \in \Re^n : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0 \quad \forall \mathbf{x} \in \mathcal{K}\}.$$

- If $\mathcal{K} = \mathbb{R}^n_+$ then we have linear optimization (LO)
- If $\mathcal{K} = SOC^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : x_0 \geq \sqrt{x_1^2 + \ldots + x_n^2}\}$ then we have second-order cone optimization(SOCO)
- If $\mathcal{K} = \mathcal{S}^n_+$ then we have (positive) semidefinite optimization (SDO)

These cones are self-dual: $\mathcal{K} = \mathcal{K}^*$.

"LO $\subsetneq$ SOCO $\subsetneq$ SDO"

The SOC constraint can be rewritten as

$$x_0^2 - x_1^2 - \ldots - x_n^2 \geq 0, x_0 \geq 0$$

and is therefore equivalent to the positive semidefinite constraint

$$\begin{pmatrix} x_0 & & & & x_1 \\ & x_0 & & & x_2 \\ & & x_0 & & x_3 \\ & & & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_0 \end{pmatrix} \succeq 0$$

where $\succeq 0$ denotes positive semidefiniteness.

Hence, SOCO is a special case of SDO, and
LO is a special case of SOCO.

Trending Now...

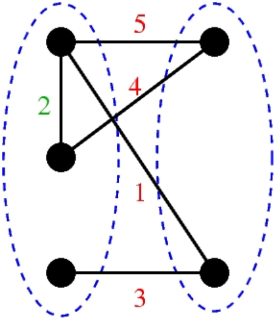
1. Increasing degree of the functions describing the models:
   - Linear
   - Quadratic
   - Polynomial
2. Increasing complexity of the convex cones underlying the models:
   - Non-negative orthant
   - Second-order and semidefinite cones
   - Copositive and completely positive cones
3. Increasing variety of algorithms to compute with these cones:
   - Ellipsoid method
   - Interior-point methods
   - First-order methods; bundle methods; Lagrangian methods
   - Combinations with branch-and-bound, cutting planes
4. Increasing array of applications:
   - Control theory
   - Combinatorial optimization
   - Finance
   - Engineering
   - Polynomial optimization

Trending Now...

1. Increasing degree of the functions describing the models:
    - Linear
    - Quadratic
    - Polynomial
2. Increasing complexity of the convex cones underlying the models:
    - Non-negative orthant
    - Second-order and semidefinite cones
    - Copositive and completely positive cones
3. Increasing variety of algorithms to compute with these cones:
    - Ellipsoid method
    - Interior-point methods
    - First-order methods; bundle methods; Lagrangian methods
    - Combinations with branch-and-bound, cutting planes
4. Increasing array of applications:
    - Control theory
    - Combinatorial optimization
    - Finance
    - Engineering
    - Polynomial optimization

A Success Story: The Max-Cut Problem

Given a graph $G = (V, E)$ and weights w_{ij} for all edges $(i, j) \in E$, find an edge-cut of maximum weight, i.e. find a set $S \subseteq V$ s.t. the sum of the weights of the edges with one end in S and the other in $V \setminus S$ is maximum.



Max-Cut is in fact equivalent to quadratic unconstrained binary optimization (QUBO), and hence has many applications.

Integer Linear Optimization (ILO) Formulation

$$\begin{array}{ll} \max & \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_{ij} y_{ij} \\ \text{s.t.} & y_{ij} + y_{ik} + y_{jk} \le 2, 1 \le i < j < k \le n \\ & y_{ij} - y_{ik} - y_{jk} \le 0, 1 \le i < j \le n, k \ne i, j \\ & y_{ij} \in \{0, 1\}, 1 \le i < j \le n \end{array}$$

where

$$y_{ij} = \begin{cases} 1 & \text{if edge } ij \text{ is cut} \\ 0 & \text{otherwise,} \end{cases}$$

$y_{ij} = y_{ji}$, and w_{ij} denotes the weight of edge ij.

This formulation is the basis for a highly successful branch-and-cut algorithm for solving spin glass problems in physics (Liers, Jünger, Reinelt and Rinaldi (2005)).

Quadratic Formulation of Max-Cut

Whereas the ILO formulation is edge-based, we use a node-based quadratic formulation.

- Let the vector $v \in \{-1, +1\}^n$ represent any cut in the graph via the interpretation that the sets $\{i | v_i = +1\}$ and $\{i | v_i = -1\}$ specify the partition.
- Then max-cut may be formulated as:

$$\begin{array}{ll} \max & \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_{ij} \left(\frac{1 - v_i v_j}{2} \right) \\ \text{s.t.} & v_i^2 = 1, i = 1, \ldots, n. \end{array}$$

- If $w_{ij} = w_{ji}$ and $w_{ii} = 0$ (without loss of generality), then:

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} w_{ij} \left(\frac{1 - v_i v_j}{2} \right) = v^T Q v \text{ with } Q \text{ symmetric.}$$

The Basic Semidefinite Relaxation of Max-Cut

Consider the change of variable $X = vv^T, v \in \{\pm 1\}^n$.
Then $X_{ij} = v_i v_j$, $v^T Q v = \langle Q, vv^T \rangle = \langle Q, X \rangle$, and max-cut is equivalent to

$$\begin{array}{ll} \max & \langle Q, X \rangle \\ \text{s.t.} & \text{diag}(X) = e \\ & \text{rank}(X) = 1 \\ & X \succeq 0, \end{array}$$

where e denotes the vector of ones, and $X \succeq 0$ denotes that X is symmetric positive semidefinite.

Removing the rank constraint, we obtain the basic SDO relaxation of max-cut:

$$\begin{array}{ll} \max & \langle Q, X \rangle \\ \text{s.t.} & \text{diag}(X) = e \\ & X \succeq 0. \end{array}$$

Max-Cut Solvers Available On the Web

LO-based Spin Glass Server (Liers, Jünger, Reinelt and Rinaldi (2005):

`http://www.informatik.uni-koeln.de/spinglass/`

An SDO-based cutting-plane framework is a key ingredient of the max-cut solver *Biqmac* (Rendl, Rinaldi and Wiegele (2007)):

`http://biqmac.uni-klu.ac.at/`

A similar approach with the addition of a quadratic regularization term to the semidefinite relaxation is the basis of the solver *BiqCrunch* (Krislock, Malick and Roupin (2012)):

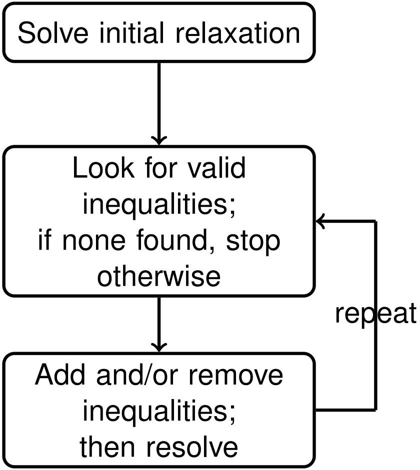
`http://lipn.univ-paris13.fr/BiqCrunch/`

Selected Extensions

This basic relaxation of max-cut is also the basis for successful solution approaches to other problems, including:

- Max-*k*-cut problems (Ghaddar, A. and Liers (2007); A., Ghaddar, Hupp, Liers, Wiegele (2013))
- Min-bisection problems (Armbruster, Helmberg, Fügenschuh and Martin (2011))
- *k*-clustering probllems (Krislock, Malick and Roupin (2014))
- Single-row facility layout problems (A., Kennings, Vannelli (2005); A. and Vannelli (2008); A. and Yen (2009); Hungerländer and Rendl (2012))
- Multi-row facility layout problems (Hungerländer and A. (2012, 2014))

Algorithmic Framework: Cutting-Plane Algorithm



For max-cut one can use:

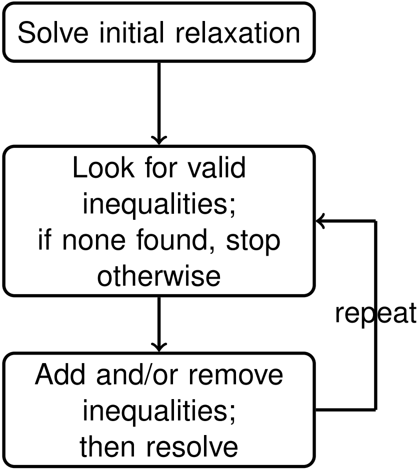
Initial Relaxation

$$\begin{array}{ll} \max & \langle Q, X\rangle \\ \text{s.t.} & \operatorname{diag}(X) = e \\ & X \succeq 0 \end{array}$$

Triangle Inequalities

$$\begin{array}{r} X_{ij} + X_{ik} + X_{jk} \geq -1 \\ -X_{ij} + X_{ik} + X_{jk} \geq -1 \\ X_{ij} - X_{ik} + X_{jk} \geq -1 \\ X_{ij} + X_{ik} - X_{jk} \geq -1 \end{array}$$

General Research Objective: Improve the Performance of this Framework



Today: Three Aspects

1. SOC relaxations for binary quadratic optimization
2. Complexity of handling large numbers of inequalities
3. Another success story: Application to row layout problems

SOC Relaxations for Binary Quadratic Optimization

This is joint work with B. Ghaddar and J.C. Vera.

B. Ghaddar, J.C. Vera, and M.F. Anjos.
Second-Order Cone Relaxations for Binary Quadratic Polynomial Programs.
SIAM Journal on Optimization, 21(1), 2011, 391-414.

Polynomial Optimization Perspective

A general polynomial optimization (PO) problem has the form:

$$\begin{array}{ll} \sup & f(x) \\ \text{s.t.} & g_i(x) \geq 0, \; i = 1, \ldots, m \end{array}$$

The problem can be equivalently written in the form:

$$\begin{array}{ll} \inf & \lambda \\ \text{s.t.} & \lambda - f(x) \geq 0 \quad \forall x \in S := \{x : g_i(x) \geq 0, \; i = 1, \ldots, m\} \end{array}$$

which we re-write as

$$\begin{array}{ll} \inf & \lambda \\ \text{s.t.} & \lambda - f(x) \in \mathcal{P}_d(S) \end{array}$$

where

$$\mathcal{P}_d(S) = \{p(x) \in \mathbf{R}_d[x] : p(s) \geq 0 \text{ for all } s \in S\}$$

is the cone of polynomials of degree $\leq d$ that are non-negative over S.

A General Recipe for Relaxations of PO

The set

$$\mathcal{P}_d(S) = \{p(x) \in \mathbf{R}_d[x] : p(s) \geq 0 \text{ for all } s \in S\}$$

is in general a very complex object.

- It is always a convex cone.
- In most cases the decision problem for $\mathcal{P}_d(S)$ is NP-hard.
- Hence the condition $\lambda - f(x) \in \mathcal{P}_d(S)$ is NP-hard in general.
- We relax it to $\lambda - f(x) \in \mathcal{K}$ for a suitable $\mathcal{K} \subseteq \mathcal{P}_d(S)$.
- Then

$$\begin{array}{ll} \inf & \lambda \\ \text{s.t.} & \lambda - f(x) \in \mathcal{K} \end{array}$$

  provides an upper bound for the original problem.
- The choice of the convex cone $\mathcal{K}$ is key for the quality of the resulting bounds.

Building Blocks for Conic Representations - I

Consider the following example:

- $p(x, y) = 8 - 2x - x^2 - 12xy + 2y^2$
- $S = \{(x, y) : x^2 + y^2 \leq 1\}$

Observe that

- $p(3, 0) < 0$, but
- $p(x, y) \geq 0$ for all $(x, y) \in S$, since

$$p(x, y) = (1 - 2x + 2y)^2 + (1 + x - 2y)^2 + 6(1 - x^2 - y^2).$$

Theorem (S-lemma)

Let $\mathcal{B} := \{x : \|x\|^2 \leq 1\}$. Then

$$p(x) \in \mathcal{P}_2(\mathcal{B}) \text{ iff } p(x) = \alpha(x) + c(1 - \|x\|^2)$$

where $\alpha(x)$ is a degree-2 sum-of-squares (SOS) and $c \geq 0$.

(see the survey of Pólik and Terlaky (2007)).

Applying the S-lemma

How to find $\alpha(x)$ and c?

Every degree-2 SOS has the form

$$\alpha(x) = \begin{bmatrix} 1 & x^{\mathrm{T}} \end{bmatrix} M \begin{bmatrix} 1 \\ x \end{bmatrix} \quad \text{with } M \succeq 0.$$

Therefore proving non-negativity of $p(x, y)$ reduces to a semidefinite feasibility problem:

$$\begin{bmatrix} 8 & -1 & 0 \\ -1 & -1 & -6 \\ 0 & -6 & 2 \end{bmatrix} = M + c \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ with } M \succeq 0, c \geq 0.$$

Corollary

Optimizing a polynomial of degree 2 over the unit ball can be cast as a conic (semidefinite) optimization problem.

Building Blocks for Conic Representations - II

- $p(x, y) = 5 - 4x + 3y$
- $S = \{(x, y) : x^2 + y^2 \leq 1\}$

Observe that $p(x, y) \geq 0$ for all $(x, y) \in S$ since

$$p(x, y) = 5 + (-4, 3)\begin{bmatrix} x \\ y \end{bmatrix} \geq 5 - \|(-4, 3)\| = 0$$

Theorem

Let $\mathcal{B} = \{x : \|x\| \leq 1\}$ then

$$p(x) \in \mathcal{P}_1(\mathcal{B}) \text{ iff } p(x) = a^{\mathrm{T}}\begin{bmatrix} 1 \\ x \end{bmatrix}, a \in \mathfrak{L} = \{(u_o, u) : \|u\| \leq u_o\}.$$

Corollary

Optimizing a polynomial of degree 1 over the unit ball can be cast as a conic (SOC) optimization problem.

What about $\mathcal{P}_3(\mathcal{B})$?

Theorem (Nesterov (2003))

The decision problem over $\mathcal{P}_3(\mathcal{B})$ is NP-hard.

In summary:

Let $\mathcal{B} = \{x : \|x\| \leq 1\}$. The decision problem $p(x) \in \mathcal{P}_d(\mathcal{B})$

- is easy for $d = 1, 2$;
- is NP-hard for $d \geq 3$.

Standard Polynomial Optimization Approach

Generate hierarchy of tractable approximations of $\mathcal{P}_d(S)$:

$$\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \cdots \subseteq \mathcal{P}_d(S)$$

First Example: The Max-Cut Problem

Consider the well-known formulation of max-cut:

$$\begin{array}{ll} \max & \frac{1}{2}\sum_{i<j} w_{ij}(1 - x_i x_j) \\ \text{s.t.} & x \in \{-1, 1\} \end{array}$$

The only constraints are the binary constraints. We recast it as:

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \lambda - \frac{1}{2}\sum_{i<j} w_{ij}(1 - x_i x_j) \in \mathcal{P}_2\left(\{-1, 1\}^n\right) \end{array}$$

Hierarchy of Relaxations

Degree-r Relaxation

$$
\begin{aligned}
(P_r)\ \min\ & \lambda \\
\text{s.t. } & \lambda - \frac{1}{2}\sum_{i<j} w_{ij}(1 - x_i x_j) = \alpha(x) + \sum_i f_i(x)(1 - x_i^2) \\
& \alpha(x) \text{ is SOS of degree } \leq r \\
& f_i(x) \text{ is polynomial of degree } \leq r-2
\end{aligned}
$$

Theorem

$$(P_1)^* \geq (P_2)^* \geq \cdots \geq (P_n)^* = (P)^*.$$

Second Example: Quadratic Knapsack Problem (QKP)

$$\begin{array}{ll} \max & q(x) = \begin{bmatrix} 1 & x \end{bmatrix} Q \begin{bmatrix} 1 \\ x \end{bmatrix} \\ \text{s.t.} & w^T x \leq c \\ & x \in \{-1, 1\}^n \end{array}$$

- Generalization of the classical linear knapsack problem
- Introduced by Gallo, Hammer and Simeone (1980)
- Known to be NP-hard

Note that QKP has the binary constraints plus one linear constraint.

PO Reformulation

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \lambda - q(x) \in \mathcal{P}_2 \left(\{-1, 1\}^n \cap \{x : c - w^T x \geq 0\} \right). \end{array}$$

Basic Semidefinite Relaxation

The semidefinite relaxation HRW4 of Helmberg-Rendl-Weismantel (2000) is equivalent to relaxing $\mathcal{P}_2(\{-1,1\}^n \cap \{x : c - w^T x \geq 0\})$ as:

$$
\begin{aligned}
\min \quad & \lambda \\
\text{s.t.} \quad & \lambda - q(x) = \alpha(x) + \sum_i c_i(1 - x_i^2) \\
& +(c - w^T x)\left\{\sum_i d_i^-(1 - x_i) + \sum_i d_i^+(1 + x_i)\right\}
\end{aligned}
$$

where $\alpha(x)$ is a degree-2 SOS, c_i are free, and $d_i^+, d_i^- \geq 0$.

- One $(n+1) \times (n+1)$ semidefinite matrix
- $2n$ non-negative vars
- n free vars
- $\sim n^2$ constraints

A New Result on SOC Representation

Observation

$x \in \{-1, 1\}^n \Rightarrow \|x\|^2 = n$

This leads to the following result:

Lemma

If $f(x)$ is a polynomial of degree 1 and $\mathcal{B}' := \{x : \|x\|^2 = n\}$, then

$$f(x) \in \mathcal{P}_1(\mathcal{B}') \text{ if and only if } f(x) = f^T \begin{pmatrix} \sqrt{n} \\ x \end{pmatrix}$$

with $f \in \mathcal{L}^{n+1}$, where $\mathcal{L}^{n+1}$ is the second-order cone.

Basic SOCO-SDO relaxation

Basic SOCO-SDO relaxation

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \lambda - q(x) = \alpha(x) + \sum_i c_i(1 - x_i^2) \\ & +(c - w^T x)f(x) + \sum_i (1 - x_i)g_i^-(x) + \sum_i (1 + x_i)g_i^+(x) \end{array}$$

where
$\alpha(x)$ is a degree-2 SOS, c_i are free, and $f(x), g_i^+(x), g_i^-(x) \in \mathcal{P}_1(\mathcal{B}')$.

- One $(n+1) \times (n+1)$ semidefinite matrix
- $2n+1$ vectors in $\mathcal{L}^n$
- n free vars
- $\sim n^2$ constraints

Theorem

$$HRW4^* \geq SOCO\text{-}SDO^* \geq QKP^*$$

A Pure SOC Relaxation for QKP

To verify how much SOCO alone can do, we remove the SDO part:

Basic SOCO~~-SDO~~ relaxation

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \lambda - q(x) = \cancel{\alpha(x)} + \sum_i c_i(1 - x_i^2) \\ & +(c - w^T x)f(x) + \sum_i (1 - x_i)g_i^-(x) + \sum_i (1 + x_i)g_i^+(x) \end{array}$$

where

~~$\alpha(x)$ is a degree-2 SOS,~~ c_i are free, and $f(x), g_i^+(x), g_i^-(x) \in \mathcal{P}_1(\mathcal{B}')$.

- ~~One $(n+1) \times (n+1)$ semidefinite matrix~~
- $2n + 1$ vectors in $\mathcal{L}^n$
- n free vars
- $\sim n^2$ constraints

Comparison of Upper Bounds for the QKP

n_d_rnd	HRW4	SOCO-SDO	SOCO
50_10_50	2412.5	2353.9	2846.1
50_30_50	11485.6	11433.2	12050.9
50_50_50	23863.0	23846.1	23851.0
50_70_50	32626.5	32571.1	32575.1
50_90_50	17682.6	17671.0	17672.8
60_10_60	7216.0	7188.7	7410.1
60_30_60	26530.8	26496.5	26502.7
60_50_60	13895.5	13871.4	14396.6
60_70_60	56583.5	56561.2	56561.4
60_90_60	62015.6	62009.0	62009.0
70_10_70	4109.6	4036.7	5104.2
70_30_70	20275.1	20208.6	21826.8
70_50_70	45573.2	45507.1	45752.8
70_70_70	1882.8	1631.6	1737.9
70_90_70	32914.0	32857.3	32876.1

Triangle Inequalities Relaxation

Any relaxation for the QKP can be improved using the triangle inequalities:

$$x \in \{-1,1\}^n \Rightarrow \begin{cases} x_i x_j + x_j x_k + x_k x_i \geq -1, \\ x_i x_j - x_j x_k - x_k x_i \geq -1, \\ -x_i x_j + x_j x_k - x_k x_i \geq -1, \\ -x_i x_j - x_j x_k + x_k x_i \geq -1. \end{cases}$$

Improved SOCO Relaxation

SOCO-△ Relaxation

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \lambda - q(x) = \sum_i c_i(1 - x_i^2) \\ & +(c - w^T x) f(x) + \textstyle\sum_i (1 - x_i) g_i^-(x) + \sum_i (1 + x_i) g_i^+(x) \\ & + \textstyle\sum_{i<j<k} d_{ijk}^{\pm,\pm}(x_i x_j \pm x_j x_k \pm x_k x_i + 1) \end{aligned}$$

where $\alpha(x)$ is a degree-2 SOS, c_i are free, $f(x), g_i^+(x), g_i^-(x) \in \mathcal{P}_1(\mathcal{B}')$, and $d_{ijk}^{\pm,\pm} \geq 0$.

- $2n + 1$ vectors in $\mathcal{L}^n$
- n free variables
- $\sim n^2$ constraints
- $\sim n^3$ nonnegative variables

Comparison of Upper Bounds for the QKP (ctd)

n_d_rnd	HRW4	SOCO-SDO	SOCO	SOCO-SDO-△	SOCO-△
50_10_50	2412.5	2353.9	2846.1	2316.1	2316.1
50_30_50	11485.6	11433.2	12050.9	11403.2	11403.2
50_50_50	23863.0	23846.1	23851.0	23815.1	23815.8
50_70_50	32626.5	32571.1	32575.1	32555.5	32555.5
50_90_50	17682.6	17671.0	17672.8	17651.5	17651.6
60_10_60	7216.0	7188.7	7410.1	7170.7	7170.8
60_30_60	26530.8	26496.5	26502.7	26488.8	26488.9
60_50_60	13895.5	13871.4	14396.6	13844.5	13845.7
60_70_60	56583.5	56561.2	56561.4	56552.6	56552.6
60_90_60	62015.6	62009.0	62009.0	62009.0	62009.0
70_10_70	4109.6	4036.7	5104.2	3957.4	3958.1
70_30_70	20275.1	20208.6	21826.8	20190.0	20190.0
70_50_70	45573.2	45507.1	45752.8	45488.1	45488.2
70_70_70	1882.8	1631.6	1737.9	1619.7	1619.6
70_90_70	32914.0	32857.3	32876.1	32837.2	32837.2

Comparison of Computational Time (sec)

n_d_rnd	HRW4	SOCO-SDO	SOCO	SOCO-SDO-△	SOCO-△
50_10_50	121.1	122.4	10.9	755.3	386.9
50_30_50	122.8	134.6	11.2	688.4	326.9
50_50_50	116.5	122.7	13.7	727.5	320.3
50_70_50	94.1	139.5	15.3	601.0	260.1
50_90_50	99.9	128.4	14.0	723.6	381.0
60_10_60	540.8	670.4	25.8	3769.0	1336.0
60_30_60	572.3	644.9	26.1	5420.0	1796.0
60_50_60	564.4	703.3	25.2	3597.0	1344.0
60_70_60	540.7	868.2	28.3	3645.0	1502.0
60_90_60	556.2	520.0	23.3	1815.0	450.2
70_10_70	2074.0	2295.0	43.8	12460.0	2192.7
70_30_70	2124.0	2360.0	48.5	9209.0	1747.0
70_50_70	2107.0	2806.0	56.2	10260.0	3212.0
70_70_70	2453.0	1631.6	46.4	11780.0	4226.0
70_90_70	2137.0	2716.0	58.4	9840.0	2825.0

In Summary

Working in the PO framework and for binary quadratic optimization, we obtained

- A tight SOCO-SDO-based relaxation.
- A cheaper pure SOCO relaxation that is competitive with SDO-relaxations.
- The impact of these relaxations was computationally demonstrated on the QKP.

It is also possible to generate polynomial cuts:

- Instead of using the classical approach to PO of approximating uniformly the whole feasible set
- apply a local approach, i.e., locally approximate the feasible set by adding valid polynomial inequalities.

Such a cut generating procedure was proposed by Ghaddar, Vera and A. (2011) for general PO, and also specialized for binary quadratic optimization.

Complexity of an Integrated Interior-Point Method with Cutting Planes

This is joint work with A. Engau.

A. Engau and M.F. Anjos.
A Primal-Dual Interior-Point Algorithm for Linear Programming with Selective Addition of Inequalities.
Cahier du GERAD G-2011-44, 2011.

Handling a Large Number of Inequalities

The question of how to add inequalities efficiently has received attention for some time.

Among the contributions in this area:

- Column-generation and cutting-plane methods (see e.g. Mitchell (2003), Mitchell (2009));
- Dual approaches, such as augmented Lagrangian relaxations (see e.g. Güler (1992), Evtushenko (2005));
- Build-up or build-down approaches for constraint-reduction in linear optimization (see e.g Dantzig-Ye (1991), Kaliski-Ye (1993), den Hertog-Roos-Terlaky (1994)),
- and more recently in convex quadratic optimization (Jung-O'Leary-Tits (2012)) and semidefinite optimization (Park (2011)).

Handling a Large Number of Inequalities (ctd)

We restrict our attention here to the class of interior-point methods (IPMs) that combine a build-up cutting-plane approach with the dynamic addition of inequalities to solve linear and/or semidefinite optimization problems, such as in:

- Helmberg-Rendl (1998)
- Mitchell (2000)
- Gruber-Rendl (2003)
- Engau-A.-Vannelli (2012)
- Engau-A.-Bomze (2013)

See the chapter by Engau:
Recent Progress in Interior-Point Methods: Cutting-Plane Algorithms and Warm Starts
in the
Handbook on Semidefinite, Conic and Polynomial Optimization.

Integrated Algorithm (Engau, A. and Vannelli (2012))

Start from the initial relaxation without any cuts $\mathcal{P}(X) \geq q$

$$\begin{aligned} \min\ & C \bullet X \\ \text{s.t. } & \mathcal{A}X = b,\ X \succeq 0 \\ & \mathcal{P}_I X - r = q_I,\ r = \xi,\ \xi \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & b^T y + q_I^T z \\ \text{s.t. } & S = C - \mathcal{A}^T y - \mathcal{P}_I^T z \succeq 0 \\ & z = \psi,\ \psi \geq 0 \end{aligned}$$

Key Steps:

- Stops the interior-point algorithm after a few steps;
- Add and remove inequalities;
- Warmstart the interior-point algorithm by turning negative variables into free variables
  and adding the requirement that the free variables be equal to new non-negative variables;
- Use an infeasible interior-point method.

What About Polynomiality?

The well-known ideas in this algorithm (and in others too) are:

- to try to predict and add relevant inequalities before they are violated, and
- to resume the algorithm from the current iterate.

While the computational performance of these algorithms is well documented in the literature,
we know of no supporting theoretical analysis or proof of convergence and worst-case complexity for such methods.

Main objective: Obtain insights into the conditions under which an algorithm of this kind is polynomial or may be exponential.

General Worst-Case Complexity Result

Theorem (Engau and A. (2011))

Under reasonable assumptions, an algorithm of this type finds an ϵ-optimal solution in

$$O(((\kappa + \tau + 1)/\epsilon)\ell(n+\ell)^{3/2}e^{\theta/11}) \text{ iterations}$$

where n is the number of variables in the primal,
$\theta = O(\ell/\sqrt{n+\ell})$ *and*
ℓ *is the number of inequalities (out of L) that are added to the problem.*

In particular, if $\ell = O(\sqrt{n})$ or $\ell \le L = O(\sqrt{n})$, then $\theta = O(1)$ and the iteration bound is polynomial: $O(((\kappa + \tau + 1)/\epsilon)\ell(n+\ell)^{3/2})$.

Moreover, we are not able to affirm polynomial time complexity if large numbers of inequalities must be added very close to optimality.

This agrees with well-known observations in practice about IPMs.

Single- and Multi-Row Facility Layout

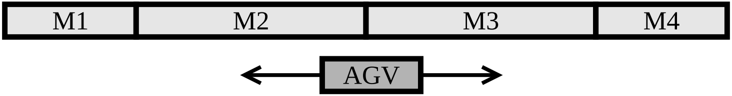
This is joint work with P. Hungerländer.

P. Hungerländer and M.F. Anjos.
A Semidefinite Optimization Approach to Space-Free Multi-Row Facility Layout.
Cahier du GERAD G-2012-03, revised May 2014.

P. Hungerländer and M.F. Anjos.
A Semidefinite Optimization-Based Approach for Global Optimization of Multi-Row Facility Layout.
Cahier du GERAD G-2014-45, 2014.

The Single-Row Facility Layout Problem (SRFLP)

The SRFLP consists in finding an optimal linear placement of departments with varying dimensions on a straight line:



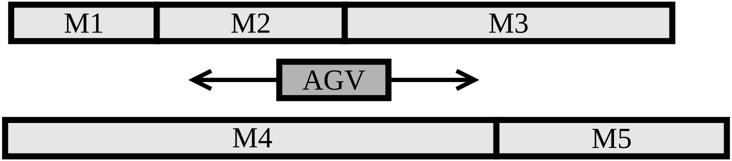
An instance of the problem consists of:

- n 1-D departments $\{1, \ldots, n\}$
- with positive lengths $\ell_1, \ldots, \ell_n$
- and (usually non-negative) pairwise connectivities c_{ij}.

We seek a placement of the departments so as to minimize the total weighted sum of the center-to-center distances between all pairs of facilities.

The Multi-Row Facility Layout Problem (MRFLP)

Similar to the SRFLP but with two of more rows available to place the departments:



Global Optimization of the SRFLP

The SRFLP can be modeled based on the quadratic formulation of max-cut (A., Kennings and Vannelli (2005)):

$$\begin{aligned} \min \quad & \sum_{i<j} \left[c_{ij} \sum_{k \neq i,j} \ell_k \left(\frac{1 - R_{ki} R_{kj}}{2} \right) \right] + \sum_{i<j} \frac{1}{2} c_{ij} (\ell_i + \ell_j) \\ \text{s.t.} \quad & R_{ij} R_{jk} - R_{ij} R_{ik} - R_{ik} R_{jk} = -1 \text{ for all triples } i < j < k \\ & R_{ij}^2 = 1 \text{ for all pairs } i < j \end{aligned}$$

where for each pair i, j of departments:

$$R_{ij} := \begin{cases} 1, & \text{if facility } i \text{ is to the right of facility } j \\ -1, & \text{if facility } i \text{ is to the left of facility } j. \end{cases}$$

Using the same algorithmic framework as for max-cut,

- A. and Vannelli (2008) solved instances with up to 30 departments to global optimality, and
- Hungerländer and Rendl (2013) computed global optima for up to 42 departments.

What about the MRFLP?

There are three additional modeling issues that arise in the MRFLP but not in the SRFLP:

1. Expressing the center-to-center distance between departments assigned to different rows;
2. Assigning each department to exactly one row;
3. Handling the possibility of empty space between departments.

If empty space is not allowed then we have the space-free MRFLP.

- The SDO model from single-row can be extended to this problem while retaining the quality of the bounds (Hungerländer and A. (2012)).

Handling space is seemingly more tricky; all the models in the literature use continuous variables to model the spaces between departments, and that significantly weakens the bounds.

Modeling Spaces in the MRFLP

Theorem

If all the department lengths are integer, then there is always an optimal solution to the MRFLP on the half grid.

Corollary

If all the department lengths are integer, then for each instance of the MRFLP we obtain an equivalent instance by adding spacing departments of length 0.5 such that the length of each row becomes equal to $\sum_{i=1}^{n} \ell_i$.

Lemma

If all departments have the same length ℓ*, then spaces of size* ℓ *are sufficient to preserve an optimal solution.*

Global Optimization of the MRFLP

Many computational results, too little time...

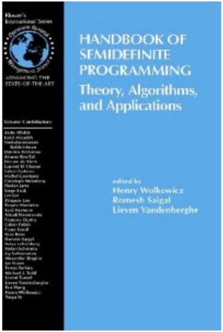
Putting all these tools together gives, to the best of our knowledge, the first global optimization approach for multi-row layout that is applicable beyond the double-row case.

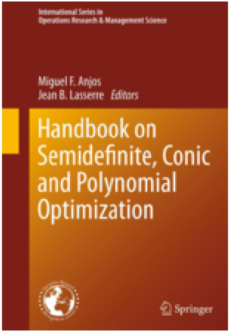
Time to wrap up...

Concluding Thoughts

- Conic optimization continues to be an exciting research area.
- Initially much attention was given to SDO; now other cones are getting more attention, for example:
  - mixed-integer SOCO
  - copositive and completely positive cones
- There remains a need to improve the performance of algorithms for conic optimization:
  1. Solve *very large* problems (large variables and many constraints): bundle methods, partial Lagrangians, etc.
  2. Better tools to exploit the structure of the problem.
- The use of conic optimization is expanding to more and more applications, and I am confident that it will remain fruitful for years to come.

Read More About It...





For papers, references, questions, you are welcome to contact me:

`miguel-f.anjos@polymtl.ca`

Thank you for your attention and enjoy the rest of conference.